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Introduction to Optimization

1.1 INTRODUCTION

Optimization is the act of obtaining the best result under given circumstances. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, *optimization* can be defined as the process of finding the conditions that give the maximum or minimum value of a function. It can be seen from Fig. 1.1 that if a point x^* corresponds to the minimum value of function $f(x)$, the same point also corresponds to the maximum value of the negative of the function, $-f(x)$. Thus without loss of generality, optimization can be taken to mean minimization since the maximum of a function can be found by seeking the minimum of the negative of the same function.

In addition, the following operations on the objective function will not change the optimum solution x^* (see Fig. 1.2):

1. Multiplication (or division) of $f(x)$ by a positive constant c.
2. Addition (or subtraction) of a positive constant c to (or from) $f(x)$.

There is no single method available for solving all optimization problems efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as *mathematical programming techniques* and are generally studied as a part of operations research. *Operations research* is a branch of mathematics concerned with the application of scientific methods and techniques to decision making problems and with establishing the best or optimal solutions. The beginnings of the subject of operations research can be traced to the early period of World War II. During the war, the British military faced the problem of allocating very scarce and limited resources (such as fighter airplanes, radars, and submarines) to several activities (deployment to numerous targets and destinations). Because there were no systematic methods available to solve resource allocation problems, the military called upon a team of mathematicians to develop methods for solving the problem in a scientific manner. The methods developed by the team were instrumental in the winning of the Air Battle by Britain. These methods, such as linear programming, which were developed as a result of research on (military) operations, subsequently became known as the methods of operations research.

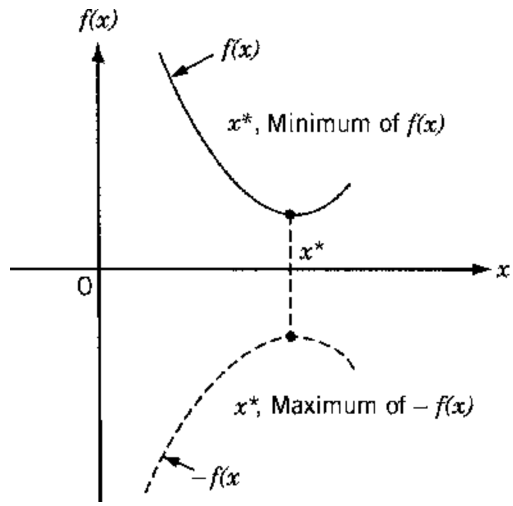

Figure 1.1 Minimum of $f(x)$ is same as maximum of $-f(x)$.

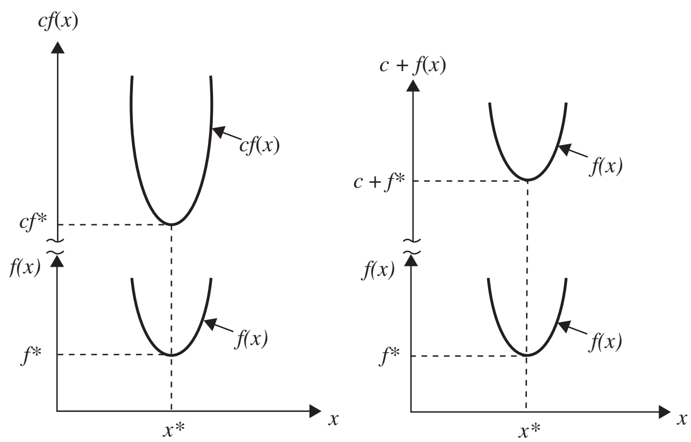

Figure 1.2 Optimum solution of $cf(x)$ or $c + f(x)$ same as that of $f(x)$.

Table 1.1 lists various mathematical programming techniques together with other well-defined areas of operations research. The classification given in Table 1.1 is not unique; it is given mainly for convenience.

Mathematical programming techniques are useful in finding the minimum of a function of several variables under a prescribed set of constraints. Stochastic process techniques can be used to analyze problems described by a set of random variables having known probability distributions. Statistical methods enable one to analyze the experimental data and build empirical models to obtain the most accurate representation of the physical situation. This book deals with the theory and application of mathematical programming techniques suitable for the solution of engineering design problems.

Table 1.1 Methods of Operations Research

Mathematical programming or optimization techniques	Stochastic process techniques	Statistical methods
Calculus methods	Statistical decision theory	Regression analysis
Calculus of variations	Markov processes	Cluster analysis, pattern recognition
Nonlinear programming	Queueing theory	
Geometric programming	Renewal theory	Design of experiments
Quadratic programming	Simulation methods	Discriminate analysis (factor analysis)
Linear programming	Reliability theory	
Dynamic programming		
Integer programming		
Stochastic programming		
Separable programming		
Multiobjective programming		
Network methods: CPM and PERT		
Game theory		
Modern or nontraditional optimization techniques		
Genetic algorithms		
Simulated annealing		
Ant colony optimization		
Particle swarm optimization		
Neural networks		
Fuzzy optimization		

1.2 HISTORICAL DEVELOPMENT

The existence of optimization methods can be traced to the days of Newton, Lagrange, and Cauchy. The development of differential calculus methods of optimization was possible because of the contributions of Newton and Leibnitz to calculus. The foundations of calculus of variations, which deals with the minimization of functionals, were laid by Bernoulli, Euler, Lagrange, and Weirstrass. The method of optimization for constrained problems, which involves the addition of unknown multipliers, became known by the name of its inventor, Lagrange. Cauchy made the first application of the steepest descent method to solve unconstrained minimization problems. Despite these early contributions, very little progress was made until the middle of the twentieth century, when high-speed digital computers made implementation of the optimization procedures possible and stimulated further research on new methods. Spectacular advances followed, producing a massive literature on optimization techniques. This advancement also resulted in the emergence of several well-defined new areas in optimization theory.

It is interesting to note that the major developments in the area of numerical methods of unconstrained optimization have been made in the United Kingdom only in the 1960s. The development of the simplex method by Dantzig in 1947 for linear programming problems and the annunciation of the principle of optimality in 1957 by Bellman for dynamic programming problems paved the way for development of the methods of constrained optimization. Work by Kuhn and Tucker in 1951 on the necessary and sufficiency conditions for the optimal solution of programming problems laid the foundations for a great deal of later research in nonlinear programming. The contributions of Zoutendijk and Rosen to nonlinear programming during the early 1960s have been significant. Although no single technique has been found to be universally applicable for nonlinear programming problems, work of Carroll and Fiacco and McCormick allowed many difficult problems to be solved by using the well-known techniques of unconstrained optimization. Geometric programming was developed in the 1960s by Duffin, Zener, and Peterson. Gomory did pioneering work in integer programming, one of the most exciting and rapidly developing areas of optimization. The reason for this is that most real-world applications fall under this category of problems. Dantzig and Charnes and Cooper developed stochastic programming techniques and solved problems by assuming design parameters to be independent and normally distributed.

The desire to optimize more than one objective or goal while satisfying the physical limitations led to the development of multiobjective programming methods. Goal programming is a well-known technique for solving specific types of multiobjective optimization problems. The goal programming was originally proposed for linear problems by Charnes and Cooper in 1961. The foundations of game theory were laid by von Neumann in 1928 and since then the technique has been applied to solve several mathematical economics and military problems. Only during the last few years has game theory been applied to solve engineering design problems.

Modern Methods of Optimization. The modern optimization methods, also sometimes called nontraditional optimization methods, have emerged as powerful and popular methods for solving complex engineering optimization problems in recent years. These methods include genetic algorithms, simulated annealing, particle swarm optimization, ant colony optimization, neural network-based optimization, and fuzzy optimization. The genetic algorithms are computerized search and optimization algorithms based on the mechanics of natural genetics and natural selection. The genetic algorithms were originally proposed by John Holland in 1975. The simulated annealing method is based on the mechanics of the cooling process of molten metals through annealing. The method was originally developed by Kirkpatrick, Gelatt, and Vecchi.

The particle swarm optimization algorithm mimics the behavior of social organisms such as a colony or swarm of insects (for example, ants, termites, bees, and wasps), a flock of birds, and a school of fish. The algorithm was originally proposed by Kennedy and Eberhart in 1995. The ant colony optimization is based on the cooperative behavior of ant colonies, which are able to find the shortest path from their nest to a food source. The method was first developed by Marco Dorigo in 1992. The neural network methods are based on the immense computational power of the nervous system to solve perceptional problems in the presence of massive amount of sensory data through its parallel processing capability. The method was originally used for optimization by Hopfield and Tank in 1985. The fuzzy optimization methods were developed to solve optimization problems involving design data, objective function, and constraints stated in imprecise form involving vague and linguistic descriptions. The fuzzy approaches for single and multiobjective optimization in engineering design were first presented by Rao in 1986.

1.3 ENGINEERING APPLICATIONS OF OPTIMIZATION

Optimization, in its broadest sense, can be applied to solve any engineering problem. Some typical applications from different engineering disciplines indicate the wide scope of the subject:

1. Design of aircraft and aerospace structures for minimum weight
2. Finding the optimal trajectories of space vehicles
3. Design of civil engineering structures such as frames, foundations, bridges, towers, chimneys, and dams for minimum cost
4. Minimum-weight design of structures for earthquake, wind, and other types of random loading
5. Design of water resources systems for maximum benefit
6. Optimal plastic design of structures
7. Optimum design of linkages, cams, gears, machine tools, and other mechanical components
8. Selection of machining conditions in metal-cutting processes for minimum production cost
9. Design of material handling equipment, such as conveyors, trucks, and cranes, for minimum cost
10. Design of pumps, turbines, and heat transfer equipment for maximum efficiency
11. Optimum design of electrical machinery such as motors, generators, and transformers
12. Optimum design of electrical networks
13. Shortest route taken by a salesperson visiting various cities during one tour
14. Optimal production planning, controlling, and scheduling
15. Analysis of statistical data and building empirical models from experimental results to obtain the most accurate representation of the physical phenomenon
16. Optimum design of chemical processing equipment and plants
17. Design of optimum pipeline networks for process industries
18. Selection of a site for an industry
19. Planning of maintenance and replacement of equipment to reduce operating costs
20. Inventory control
21. Allocation of resources or services among several activities to maximize the benefit
22. Controlling the waiting and idle times and queueing in production lines to reduce the costs
23. Planning the best strategy to obtain maximum profit in the presence of a competitor
24. Optimum design of control systems

1.4 STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\le 0, \qquad j = 1, 2, \ldots, m \\ l_j(\mathbf{X}) &= 0, \qquad j = 1, 2, \ldots, p \end{aligned} \tag{1.1}$$

where $\mathbf{X}$ is an n-dimensional vector called the *design vector*, $f(\mathbf{X})$ is termed the *objective function*, and $g_j(\mathbf{X})$ and $l_j(\mathbf{X})$ are known as *inequality* and *equality* constraints, respectively. The number of variables n and the number of constraints m and/or p need not be related in any way. The problem stated in Eq. (1.1) is called a *constrained optimization problem*.† Some optimization problems do not involve any constraints and can be stated as

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X}) \tag{1.2}$$

Such problems are called *unconstrained optimization problems*.

1.4.1 Design Vector

Any engineering system or component is defined by a set of quantities some of which are viewed as variables during the design process. In general, certain quantities are usually fixed at the outset and these are called *preassigned parameters*. All the other quantities are treated as variables in the design process and are called *design* or *decision variables* x_i, $i = 1, 2, \ldots, n$. The design variables are collectively represented as a design vector $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}^{\mathrm{T}}$. As an example, consider the design of the gear pair shown in Fig. 1.3, characterized by its face width b, number of teeth T_1 and T_2, center distance d, pressure angle ψ, tooth profile, and material. If center distance d, pressure angle ψ, tooth profile, and material of the gears are fixed in advance, these quantities can be called *preassigned parameters*. The remaining quantities can be collectively represented by a design vector $\mathbf{X} = \{x_1, x_2, x_3\}^{\mathrm{T}} = \{b, T_1, T_2\}^{\mathrm{T}}$. If there are no restrictions on the choice of b, T_1, and T_2, any set of three numbers will constitute a design for the gear pair. If an n-dimensional Cartesian space with each coordinate axis representing a design variable x_i $(i = 1, 2, \ldots, n)$ is considered, the space is called

†In the mathematical programming literature, the equality constraints $l_j(\mathbf{X}) = 0$, $j = 1, 2, \ldots, p$ are often neglected, for simplicity, in the statement of a constrained optimization problem, although several methods are available for handling problems with equality constraints.

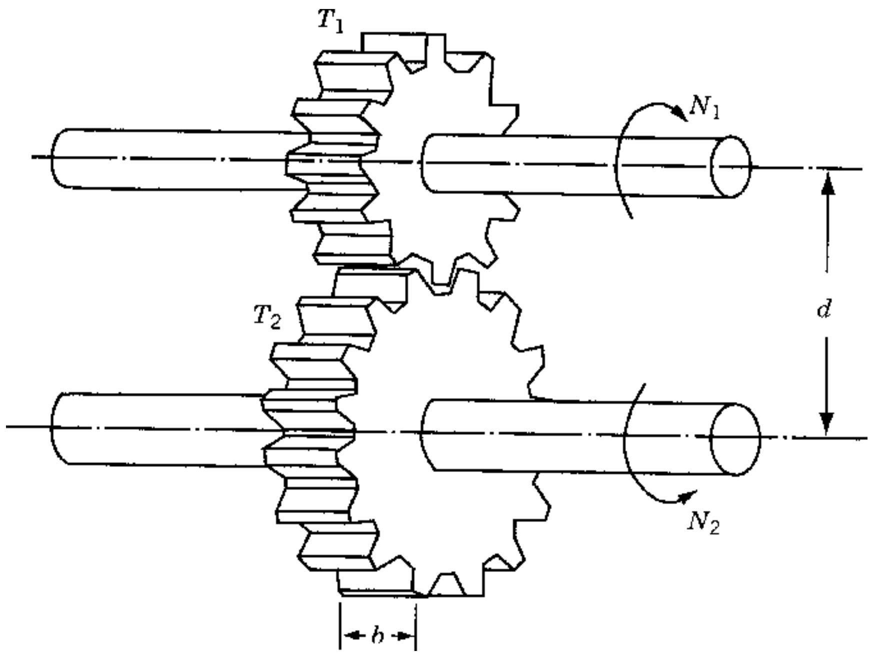

Figure 1.3 Gear pair in mesh.

the *design variable space* or simply *design space*. Each point in the n-dimensional design space is called a *design point* and represents either a possible or an impossible solution to the design problem. In the case of the design of a gear pair, the design point $\{1.0, 20, 40\}^T$, for example, represents a possible solution, whereas the design point $\{1.0, -20, 40.5\}^T$ represents an impossible solution since it is not possible to have either a negative value or a fractional value for the number of teeth.

1.4.2 Design Constraints

In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements. The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints*. Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints*. Constraints that represent physical limitations on design variables, such as availability, fabricability, and transportability, are known as *geometric* or *side constraints*. For example, for the gear pair shown in Fig. 1.3, the face width b cannot be taken smaller than a certain value, due to strength requirements. Similarly, the ratio of the numbers of teeth, T_1/T_2, is dictated by the speeds of the input and output shafts, N_1 and N_2. Since these constraints depend on the performance of the gear pair, they are called behavior constraints. The values of T_1 and T_2 cannot be any real numbers but can only be integers. Further, there can be upper and lower bounds on T_1 and T_2 due to manufacturing limitations. Since these constraints depend on the physical limitations, they are called side *constraints*.

1.4.3 Constraint Surface

For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \leq 0$. The set of values of $\mathbf{X}$ that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface*. Note that this is an $(n-1)$-dimensional subspace, where n is the number of design variables. The constraint surface divides the design space into two regions: one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0,\ j = 1, 2, \ldots, m$, which separates the acceptable region is called the *composite constraint surface*.

Figure 1.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines. A design point that lies on one or more than one constraint surface is called a *bound point*, and the associated constraint is called an *active constraint*. Design points that do not lie on any constraint surface are known as *free points*. Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

1. Free and acceptable point
2. Free and unacceptable point
3. Bound and acceptable point
4. Bound and unacceptable point

All four types of points are shown in Fig. 1.4.

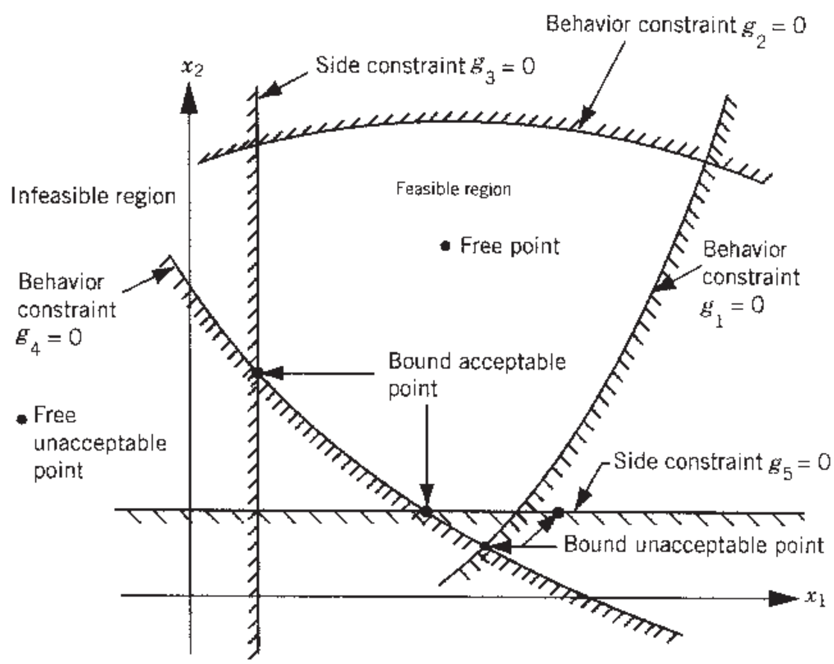

Figure 1.4 Constraint surfaces in a hypothetical two-dimensional design space.

1.4.4 Objective Function

The conventional design procedures aim at finding an acceptable or adequate design that merely satisfies the functional and other requirements of the problem. In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available. Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one. The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion* or *merit* or *objective function*. The choice of objective function is governed by the nature of problem. The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems. In civil engineering structural designs, the objective is usually taken as the minimization of cost. The maximization of mechanical efficiency is the obvious choice of an objective in mechanical engineering systems design. Thus the choice of the objective function appears to be straightforward in most design problems. However, there may be cases where the optimization with respect to a particular criterion may lead to results that may not be satisfactory with respect to another criterion. For example, in mechanical design, a gearbox transmitting the maximum power may not have the minimum weight. Similarly, in structural design, the minimum weight design may not correspond to minimum stress design, and the minimum stress design, again, may not correspond to maximum frequency design. Thus the selection of the objective function can be one of the most important decisions in the whole optimum design process.

In some situations, there may be more than one criterion to be satisfied simultaneously. For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower. An optimization problem involving multiple objective functions is known as a *multiobjective programming problem*. With multiple objectives there arises a possibility of conflict, and one simple way to handle the problem is to construct an overall objective function as a linear combination of the conflicting multiple objective functions. Thus if $f_1(\mathbf{X})$ and $f_2(\mathbf{X})$ denote two objective functions, construct a new (overall) objective function for optimization as

$$f(\mathbf{X}) = \alpha_1 f_1(\mathbf{X}) + \alpha_2 f_2(\mathbf{X}) \tag{1.3}$$

where α_1 and α_2 are constants whose values indicate the relative importance of one objective function relative to the other.

1.4.5 Objective Function Surfaces

The locus of all points satisfying $f(\mathbf{X}) = C =$ constant forms a hypersurface in the design space, and each value of C corresponds to a different member of a family of surfaces. These surfaces, called *objective function surfaces*, are shown in a hypothetical two-dimensional design space in Fig. 1.5.

Once the objective function surfaces are drawn along with the constraint surfaces, the optimum point can be determined without much difficulty. But the main problem is that as the number of design variables exceeds two or three, the constraint and objective function surfaces become complex even for visualization and the problem

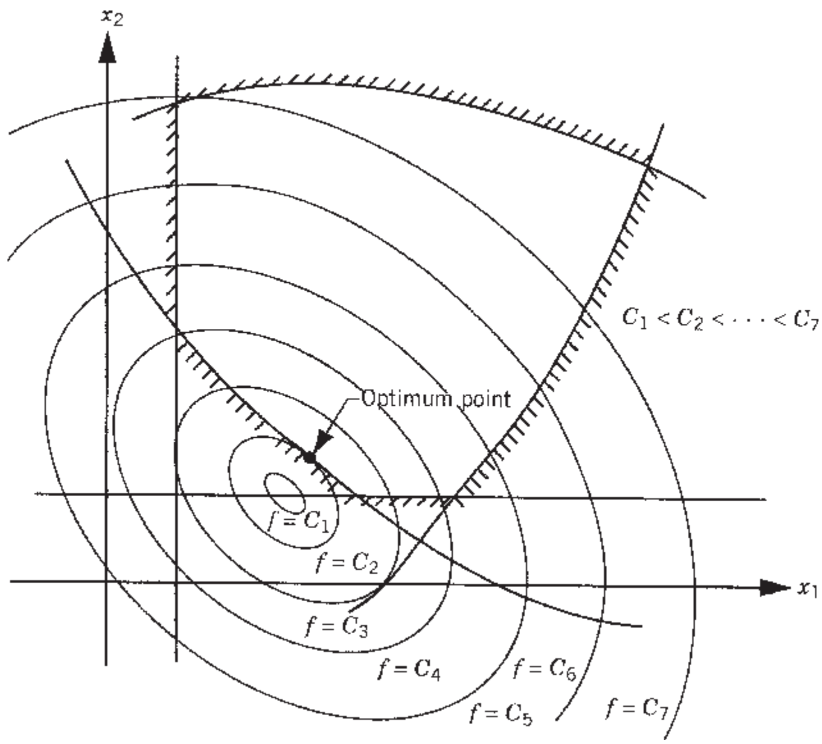

Figure 1.5 Contours of the objective function.

has to be solved purely as a mathematical problem. The following example illustrates the graphical optimization procedure.

Example 1.1 Design a uniform column of tubular section, with hinge joints at both ends, (Fig. 1.6) to carry a compressive load $P = 2500\,\text{kg}_\text{f}$ for minimum cost. The column is made up of a material that has a yield stress (σ_y) of $500\,\text{kg}_\text{f}/\text{cm}^2$, modulus of elasticity (E) of $0.85 \times 10^6\,\text{kg}_\text{f}/\text{cm}^2$, and weight density ($\rho$) of $0.0025\,\text{kg}_\text{f}/\text{cm}^3$. The length of the column is 250 cm. The stress induced in the column should be less than the buckling stress as well as the yield stress. The mean diameter of the column is restricted to lie between 2 and 14 cm, and columns with thicknesses outside the range 0.2 to 0.8 cm are not available in the market. The cost of the column includes material and construction costs and can be taken as $5W + 2d$, where W is the weight in kilograms force and d is the mean diameter of the column in centimeters.

SOLUTION The design variables are the mean diameter (d) and tube thickness (t):

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} d \\ t \end{Bmatrix} \tag{E$_1$}$$

The objective function to be minimized is given by

$$f(\mathbf{X}) = 5W + 2d = 5\rho l\pi\, dt + 2d = 9.82x_1x_2 + 2x_1 \tag{E$_2$}$$

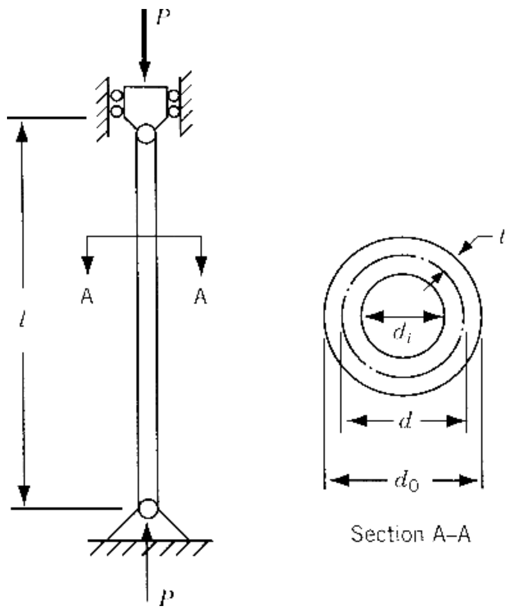

Figure 1.6 Tubular column under compression.

The behavior constraints can be expressed as

$$\text{stress induced} \le \text{yield stress}$$

$$\text{stress induced} \le \text{buckling stress}$$

The induced stress is given by

$$\text{induced stress} = \sigma_i = \frac{P}{\pi\, dt} = \frac{2500}{\pi x_1 x_2} \tag{E$_3$}$$

The buckling stress for a pin-connected column is given by

$$\text{buckling stress} = \sigma_b = \frac{\text{Euler buckling load}}{\text{cross-sectional area}} = \frac{\pi^2 EI}{l^2}\frac{1}{\pi\, dt} \tag{E$_4$}$$

where

$$\begin{aligned} I &= \text{second moment of area of the cross section of the column} \\ &= \frac{\pi}{64}(d_o^4 - d_i^4) \\ &= \frac{\pi}{64}(d_o^2 + d_i^2)(d_o + d_i)(d_o - d_i) = \frac{\pi}{64}[(d+t)^2 + (d-t)^2] \\ &\quad \times [(d+t) + (d-t)][(d+t) - (d-t)] \\ &= \frac{\pi}{8}\, dt(d^2 + t^2) = \frac{\pi}{8} x_1 x_2 (x_1^2 + x_2^2) \end{aligned} \tag{E$_5$}$$

Thus the behavior constraints can be restated as

$$g_1(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - 500 \le 0 \tag{E$_6$}$$

$$g_2(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - \frac{\pi^2(0.85 \times 10^6)(x_1^2 + x_2^2)}{8(250)^2} \le 0 \tag{E$_7$}$$

The side constraints are given by

$$2 \le d \le 14$$

$$0.2 \le t \le 0.8$$

which can be expressed in standard form as

$$g_3(\mathbf{X}) = -x_1 + 2.0 \le 0 \tag{E$_8$}$$

$$g_4(\mathbf{X}) = x_1 - 14.0 \le 0 \tag{E$_9$}$$

$$g_5(\mathbf{X}) = -x_2 + 0.2 \le 0 \tag{E$_{10}$}$$

$$g_6(\mathbf{X}) = x_2 - 0.8 \le 0 \tag{E$_{11}$}$$

Since there are only two design variables, the problem can be solved graphically as shown below.

First, the constraint surfaces are to be plotted in a two-dimensional design space where the two axes represent the two design variables x_1 and x_2. To plot the first constraint surface, we have

$$g_1(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - 500 \le 0$$

that is,

$$x_1 x_2 \ge 1.593$$

Thus the curve $x_1 x_2 = 1.593$ represents the constraint surface $g_1(\mathbf{X}) = 0$. This curve can be plotted by finding several points on the curve. The points on the curve can be found by giving a series of values to x_1 and finding the corresponding values of x_2 that satisfy the relation $x_1 x_2 = 1.593$:

x_1	2.0	4.0	6.0	8.0	10.0	12.0	14.0
x_2	0.7965	0.3983	0.2655	0.1990	0.1593	0.1328	0.1140

These points are plotted and a curve $P_1 Q_1$ passing through all these points is drawn as shown in Fig. 1.7, and the infeasible region, represented by $g_1(\mathbf{X}) > 0$ or $x_1 x_2 < 1.593$, is shown by hatched lines.† Similarly, the second constraint $g_2(\mathbf{X}) \le 0$ can be expressed as $x_1 x_2(x_1^2 + x_2^2) \ge 47.3$ and the points lying on the constraint surface $g_2(\mathbf{X}) = 0$ can be obtained as follows for $x_1 x_2(x_1^2 + x_2^2) = 47.3$:

†The infeasible region can be identified by testing whether the origin lies in the feasible or infeasible region.

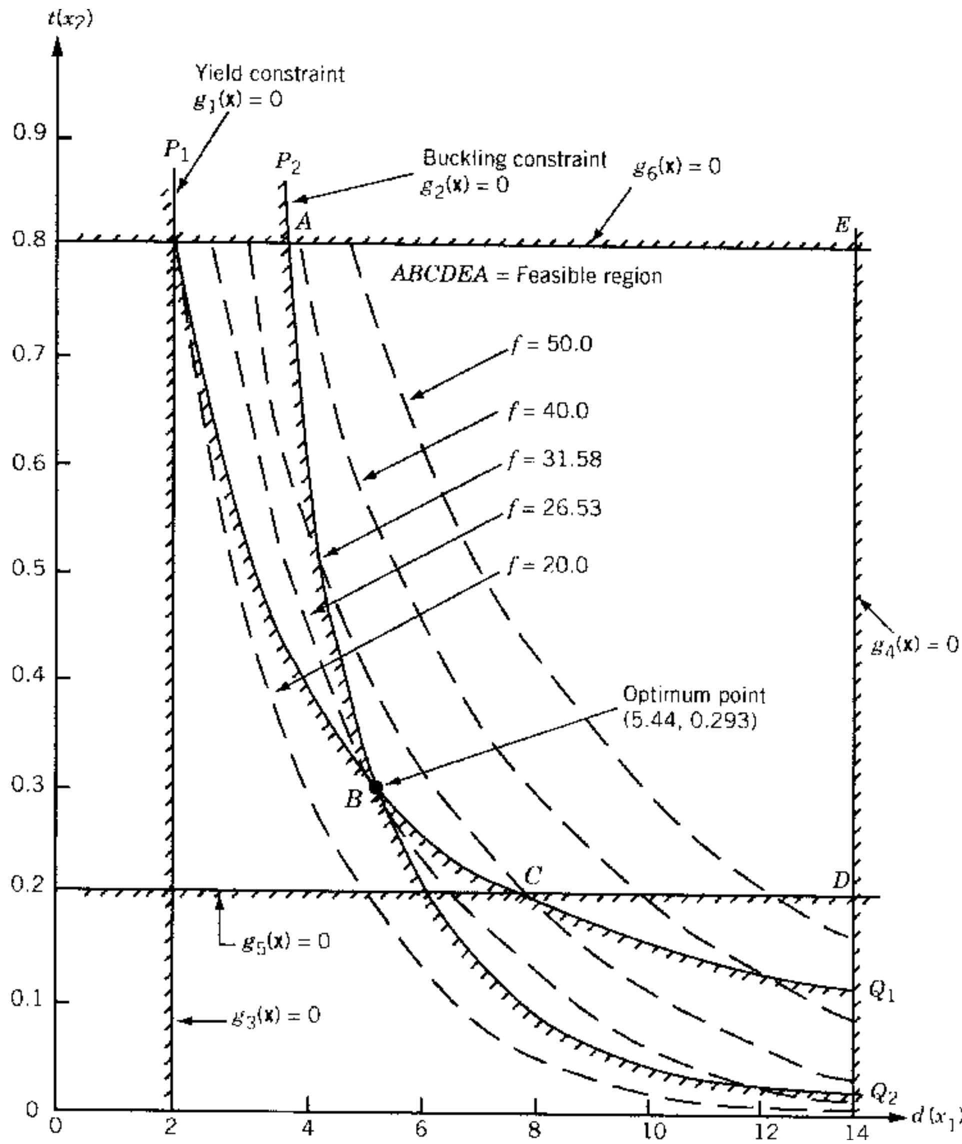

Figure 1.7 Graphical optimization of Example 1.1.

x_1	2	4	6	8	10	12	14
x_2	2.41	0.716	0.219	0.0926	0.0473	0.0274	0.0172

These points are plotted as curve P_2Q_2, the feasible region is identified, and the infeasible region is shown by hatched lines as in Fig. 1.7. The plotting of side constraints is very simple since they represent straight lines. After plotting all the six constraints, the feasible region can be seen to be given by the bounded area *ABCDEA*.

Next, the contours of the objective function are to be plotted before finding the optimum point. For this, we plot the curves given by

$$f(\mathbf{X}) = 9.82x_1x_2 + 2x_1 = c = \text{constant}$$

for a series of values of c. By giving different values to c, the contours of f can be plotted with the help of the following points.

For $9.82x_1x_2 + 2x_1 = 50.0$:

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	16.77	12.62	10.10	8.44	7.24	6.33	5.64	5.07

For $9.82x_1x_2 + 2x_1 = 40.0$:

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	13.40	10.10	8.08	6.75	5.79	5.06	4.51	4.05

For $9.82x_1x_2 + 2x_1 = 31.58$ (passing through the corner point C):

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	10.57	7.96	6.38	5.33	4.57	4.00	3.56	3.20

For $9.82x_1x_2 + 2x_1 = 26.53$ (passing through the corner point B):

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	8.88	6.69	5.36	4.48	3.84	3.36	2.99	2.69

For $9.82x_1x_2 + 2x_1 = 20.0$:

x_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_1	6.70	5.05	4.04	3.38	2.90	2.53	2.26	2.02

These contours are shown in Fig. 1.7 and it can be seen that the objective function cannot be reduced below a value of 26.53 (corresponding to point B) without violating some of the constraints. Thus the optimum solution is given by point B with $d^* = x_1^* = 5.44$ cm and $t^* = x_2^* = 0.293$ cm with $f_{\min} = 26.53$.

1.5 CLASSIFICATION OF OPTIMIZATION PROBLEMS

Optimization problems can be classified in several ways, as described below.

1.5.1 Classification Based on the Existence of Constraints

As indicated earlier, any optimization problem can be classified as constrained or unconstrained, depending on whether constraints exist in the problem.

1.5.2 Classification Based on the Nature of the Design Variables

Based on the nature of design variables encountered, optimization problems can be classified into two broad categories. In the first category, the problem is to find values to a set of design parameters that make some prescribed function of these parameters minimum subject to certain constraints. For example, the problem of minimum-weight design of a prismatic beam shown in Fig. 1.8*a* subject to a limitation on the maximum deflection can be stated as follows:

$$\text{Find } \mathbf{X} = \begin{Bmatrix} b \\ d \end{Bmatrix} \text{ which minimizes} \qquad f(\mathbf{X}) = \rho l b d \tag{1.4}$$

subject to the constraints

$$\delta_{\text{tip}}(\mathbf{X}) \le \delta_{\max}$$
$$b \ge 0$$
$$d \ge 0$$

where ρ is the density and δ_{tip} is the tip deflection of the beam. Such problems are called *parameter* or *static optimization problems*. In the second category of problems, the objective is to find a set of design parameters, which are all continuous functions of some other parameter, that minimizes an objective function subject to a set of constraints. If the cross-sectional dimensions of the rectangular beam are allowed to vary along its length as shown in Fig. 1.8*b*, the optimization problem can be stated as

$$\text{Find } \mathbf{X}(t) = \begin{Bmatrix} b(t) \\ d(t) \end{Bmatrix} \text{ which minimizes}$$
$$f[\mathbf{X}(t)] = \rho \int_0^l b(t)\; d(t)\, dt \tag{1.5}$$

subject to the constraints

$$\delta_{\text{tip}}[\mathbf{X}(t)] \le \delta_{\max}, \qquad 0 \le t \le l$$
$$b(t) \ge 0, \qquad 0 \le t \le l$$
$$d(t) \ge 0, \qquad 0 \le t \le l$$

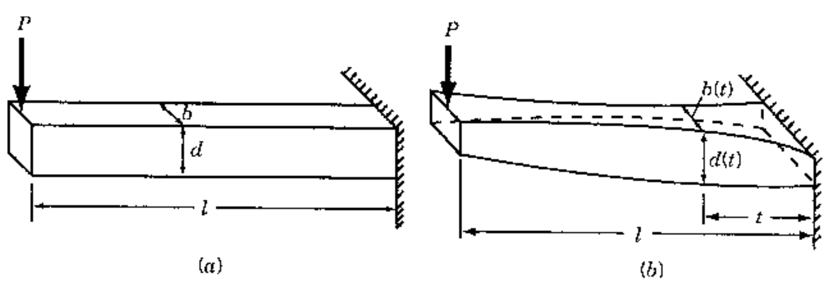

Figure 1.8 Cantilever beam under concentrated load.

Here the design variables are functions of the length parameter t. This type of problem, where each design variable is a function of one or more parameters, is known as a *trajectory* or *dynamic optimization problem* [1.55].

1.5.3 Classification Based on the Physical Structure of the Problem

Depending on the physical structure of the problem, optimization problems can be classified as optimal control and nonoptimal control problems.

Optimal Control Problem. An *optimal control* (OC) *problem* is a mathematical programming problem involving a number of stages, where each stage evolves from the preceding stage in a prescribed manner. It is usually described by two types of variables: the control (design) and the state variables. The *control variables* define the system and govern the evolution of the system from one stage to the next, and the *state variables* describe the behavior or status of the system in any stage. The problem is to find a set of control or design variables such that the total objective function (also known as the *performance index*, PI) over all the stages is minimized subject to a set of constraints on the control and state variables. An OC problem can be stated as follows [1.55]:

$$\text{Find } \mathbf{X} \text{ which minimizes } f(\mathbf{X}) = \sum_{i=1}^{l} f_i(x_i, y_i) \tag{1.6}$$

subject to the constraints

$$\begin{aligned} q_i(x_i, y_i) + y_i &= y_{i+1}, & i &= 1, 2, \ldots, l \\ \mathbf{g}_j(x_j) &\le 0, & j &= 1, 2, \ldots, l \\ \mathbf{h}_k(y_k) &\le 0, & k &= 1, 2, \ldots, l \end{aligned}$$

where x_i is the ith control variable, y_i the ith state variable, and f_i the contribution of the ith stage to the total objective function; $\mathbf{g}_j$, $\mathbf{h}_k$, and q_i are functions of x_j, y_k, and x_i and y_i, respectively, and l is the total number of stages. The control and state variables x_i and y_i can be vectors in some cases. The following example serves to illustrate the nature of an optimal control problem.

Example 1.2 A rocket is designed to travel a distance of $12s$ in a vertically upward direction [1.39]. The thrust of the rocket can be changed only at the discrete points located at distances of $0, s, 2s, 3s, \ldots, 12s$. If the maximum thrust that can be developed at point i either in the positive or negative direction is restricted to a value of F_i, formulate the problem of minimizing the total time of travel under the following assumptions:

1. The rocket travels against the gravitational force.
2. The mass of the rocket reduces in proportion to the distance traveled.
3. The air resistance is proportional to the velocity of the rocket.

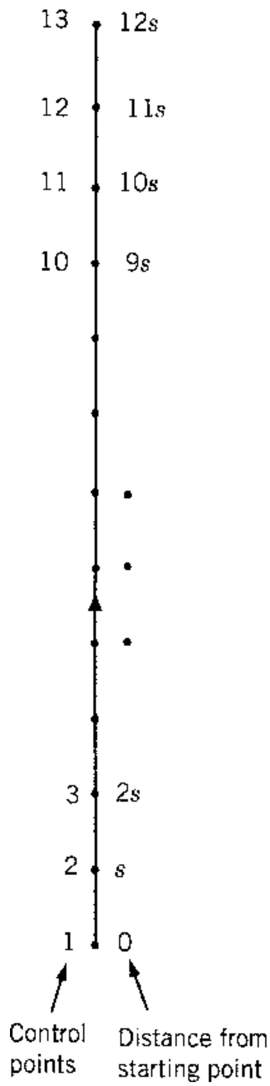

Figure 1.9 Control points in the path of the rocket.

SOLUTION Let points (or control points) on the path at which the thrusts of the rocket are changed be numbered as 1, 2, 3, . . . , 13 (Fig. 1.9). Denoting x_i as the thrust, v_i the velocity, a_i the acceleration, and m_i the mass of the rocket at point i, Newton's second law of motion can be applied as

$$\text{net force on the rocket} = \text{mass} \times \text{acceleration}$$

This can be written as

$$\text{thrust} - \text{gravitational force} - \text{air resistance} = \text{mass} \times \text{acceleration}$$

or

$$x_i - m_i g - k_1 v_i = m_i a_i \tag{E$_1$}$$

where the mass m_i can be expressed as

$$m_i = m_{i-1} - k_2 s \tag{E$_2$}$$

and k_1 and k_2 are constants. Equation (E$_1$) can be used to express the acceleration, a_i, as

$$a_i = \frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i} \tag{E$_3$}$$

If t_i denotes the time taken by the rocket to travel from point i to point $i+1$, the distance traveled between the points i and $i+1$ can be expressed as

$$s = v_i t_i + \tfrac{1}{2} a_i t_i^2$$

or

$$\frac{1}{2} t_i^2 \left(\frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i} \right) + t_i v_i - s = 0 \tag{E$_4$}$$

from which t_i can be determined as

$$t_i = \frac{-v_i \pm \sqrt{v_i^2 + 2s \left(\dfrac{x_i}{m_i} - g - \dfrac{k_1 v_i}{m_i} \right)}}{\dfrac{x_i}{m_i} - g - \dfrac{k_1 v_i}{m_i}} \tag{E$_5$}$$

Of the two values given by Eq. (E$_5$), the positive value has to be chosen for t_i. The velocity of the rocket at point $i+1$, v_{i+1}, can be expressed in terms of v_i as (by assuming the acceleration between points i and $i+1$ to be constant for simplicity)

$$v_{i+1} = v_i + a_i t_i \tag{E$_6$}$$

The substitution of Eqs. (E$_3$) and (E$_5$) into Eq. (E$_6$) leads to

$$v_{i+1} = \sqrt{v_i^2 + 2s \left(\frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i} \right)} \tag{E$_7$}$$

From an analysis of the problem, the control variables can be identified as the thrusts, x_i, and the state variables as the velocities, v_i. Since the rocket starts at point 1 and stops at point 13,

$$v_1 = v_{13} = 0 \tag{E$_8$}$$

Thus the problem can be stated as an OC problem as

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{12} \end{Bmatrix} \text{ which minimizes}$$

$$f(\mathbf{X}) = \sum_{i=1}^{12} t_i = \sum_{i=1}^{12} \left\{ \frac{-v_i + \sqrt{v_i^2 + 2s\left(\dfrac{x_i}{m_i} - g - \dfrac{k_1 v_i}{m_i}\right)}}{\dfrac{x_i}{m_i} - g - \dfrac{k_1 v_i}{m_i}} \right\}$$

subject to

$$m_{i+1} = m_i - k_2 s, \qquad i = 1, 2, \ldots, 12$$

$$v_{i+1} = \sqrt{v_i^2 + 2s\left(\frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i}\right)}, \qquad i = 1, 2, \ldots, 12$$

$$|x_i| \le F_i, \qquad i = 1, 2, \ldots, 12$$

$$v_1 = v_{13} = 0$$

1.5.4 Classification Based on the Nature of the Equations Involved

Another important classification of optimization problems is based on the nature of expressions for the objective function and the constraints. According to this classification, optimization problems can be classified as linear, nonlinear, geometric, and quadratic programming problems. This classification is extremely useful from the computational point of view since there are many special methods available for the efficient solution of a particular class of problems. Thus the first task of a designer would be to investigate the class of problem encountered. This will, in many cases, dictate the types of solution procedures to be adopted in solving the problem.

Nonlinear Programming Problem. If any of the functions among the objective and constraint functions in Eq. (1.1) is nonlinear, the problem is called a *nonlinear programming* (NLP) *problem*. This is the most general programming problem and all other problems can be considered as special cases of the NLP problem.

Example 1.3 The step-cone pulley shown in Fig. 1.10 is to be designed for transmitting a power of at least 0.75 hp. The speed of the input shaft is 350 rpm and the output speed requirements are 750, 450, 250, and 150 rpm for a fixed center distance of a between the input and output shafts. The tension on the tight side of the belt is to be kept more than twice that on the slack side. The thickness of the belt is t and the coefficient of friction between the belt and the pulleys is μ. The stress induced in the belt due to tension on the tight side is s. Formulate the problem of finding the width and diameters of the steps for minimum weight.

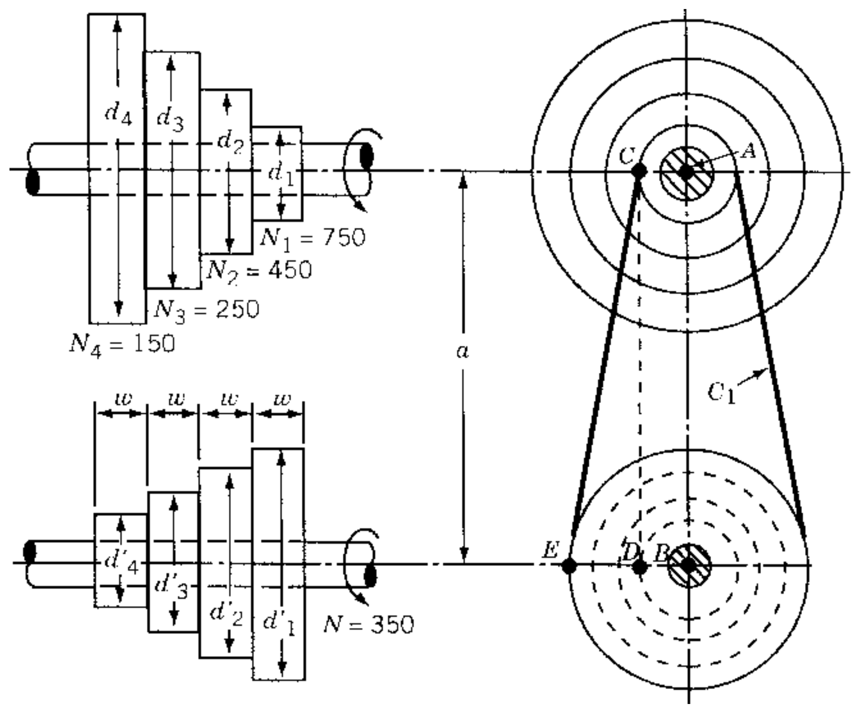

Figure 1.10 Step-cone pulley.

SOLUTION The design vector can be taken as

$$\mathbf{X} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ w \end{Bmatrix}$$

where d_i is the diameter of the ith step on the output pulley and w is the width of the belt and the steps. The objective function is the weight of the step-cone pulley system:

$$\begin{aligned} f(\mathbf{X}) &= \rho w \frac{\pi}{4}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_1'^2 + d_2'^2 + d_3'^2 + d_4'^2) \\ &= \rho w \frac{\pi}{4} \left\{ d_1^2 \left[1 + \left(\frac{750}{350} \right)^2 \right] + d_2^2 \left[1 + \left(\frac{450}{350} \right)^2 \right] \right. \\ &\quad \left. + \, d_3^2 \left[1 + \left(\frac{250}{350} \right)^2 \right] + d_4^2 \left[1 + \left(\frac{150}{350} \right)^2 \right] \right\} \end{aligned} \tag{E$_1$}$$

where ρ is the density of the pulleys and d_i' is the diameter of the ith step on the input pulley.

To have the belt equally tight on each pair of opposite steps, the total length of the belt must be kept constant for all the output speeds. This can be ensured by satisfying the following equality constraints:

$$C_1 - C_2 = 0 \tag{E$_2$}$$

$$C_1 - C_3 = 0 \tag{E$_3$}$$

$$C_1 - C_4 = 0 \tag{E$_4$}$$

where C_i denotes length of the belt needed to obtain output speed N_i ($i = 1, 2, 3, 4$) and is given by [1.116, 1.117]:

$$C_i \simeq \frac{\pi d_i}{2}\left(1 + \frac{N_i}{N}\right) + \frac{\left(\frac{N_i}{N} - 1\right)^2 d_i^2}{4a} + 2a$$

where N is the speed of the input shaft and a is the center distance between the shafts. The ratio of tensions in the belt can be expressed as [1.116, 1.117]

$$\frac{T_1^{\,i}}{T_2^{\,i}} = e^{\mu\theta_i}$$

where $T_1^{\,i}$ and $T_2^{\,i}$ are the tensions on the tight and slack sides of the ith step, μ the coefficient of friction, and θ_i the angle of lap of the belt over the ith pulley step. The angle of lap is given by

$$\theta_i = \pi - 2\sin^{-1}\left[\frac{\left(\frac{N_i}{N} - 1\right) d_i}{2a}\right]$$

and hence the constraint on the ratio of tensions becomes

$$\exp\left\{\mu\left[\pi - 2\sin^{-1}\left\{\left(\frac{N_i}{N} - 1\right)\frac{d_i}{2a}\right\}\right]\right\} \geq 2, \qquad i = 1, 2, 3, 4 \tag{E$_5$}$$

The limitation on the maximum tension can be expressed as

$$T_1^{\,i} = stw, \qquad i = 1, 2, 3, 4 \tag{E$_6$}$$

where s is the maximum allowable stress in the belt and t is the thickness of the belt. The constraint on the power transmitted can be stated as (using lb_f for force and ft for linear dimensions)

$$\frac{(T_1^{\,i} - T_2^{\,i})\pi d_i'(350)}{33{,}000} \geq 0.75$$

which can be rewritten, using $T_1^{\,i} = stw$ from Eq. (E$_6$), as

$$stw\left(1 - \exp\left[-\mu\left(\pi - 2\sin^{-1}\left\{\left(\frac{N_i}{N} - 1\right)\frac{d_i}{2a}\right\}\right)\right]\right)\pi d_i' \times \left(\frac{350}{33{,}000}\right) \geq 0.75, \qquad i = 1, 2, 3, 4 \tag{E$_7$}$$

Finally, the lower bounds on the design variables can be taken as

$$w \geq 0 \tag{E$_8$}$$

$$d_i \geq 0, \quad i = 1, 2, 3, 4 \tag{E$_9$}$$

As the objective function, (E$_1$), and most of the constraints, (E$_2$) to (E$_9$), are nonlinear functions of the design variables d_1, d_2, d_3, d_4, and w, this problem is a nonlinear programming problem.

Geometric Programming Problem.

Definition A function $h(\mathbf{X})$ is called a *posynomial* if h can be expressed as the sum of power terms each of the form

$$c_i x_1^{ai1} x_2^{ai2} \cdots x_n^{ain}$$

where c_i and a_{ij} are constants with $c_i > 0$ and $x_j > 0$. Thus a posynomial with N terms can be expressed as

$$h(\mathbf{X}) = c_1 x_1^{a11} x_2^{a12} \cdots x_n^{a1n} + \cdots + c_N x_1^{aN1} x_2^{aN2} \cdots x_n^{aNn} \tag{1.7}$$

A *geometric programming* (GMP) *problem* is one in which the objective function and constraints are expressed as posynomials in $\mathbf{X}$. Thus GMP problem can be posed as follows [1.59]:

Find $\mathbf{X}$ which minimizes

$$f(\mathbf{X}) = \sum_{i=1}^{N_0} c_i \left(\prod_{j=1}^{n} x_j^{p_{ij}} \right), \quad c_i > 0, \quad x_j > 0 \tag{1.8}$$

subject to

$$g_k(\mathbf{X}) = \sum_{i=1}^{N_k} a_{ik} \left(\prod_{j=1}^{n} x_j^{q_{ijk}} \right) > 0, \quad a_{ik} > 0, \ x_j > 0, \ k = 1, 2, \ldots, m$$

where N_0 and N_k denote the number of posynomial terms in the objective and kth constraint function, respectively.

Example 1.4 Four identical helical springs are used to support a milling machine weighing 5000 lb. Formulate the problem of finding the wire diameter (d), coil diameter (D), and the number of turns (N) of each spring (Fig. 1.11) for minimum weight by limiting the deflection to 0.1 in. and the shear stress to 10,000 psi in the spring. In addition, the natural frequency of vibration of the spring is to be greater than 100 Hz. The stiffness of the spring (k), the shear stress in the spring (τ), and the natural frequency of vibration of the spring (f_n) are given by

$$k = \frac{d^4 G}{8 D^3 N}$$

$$\tau = K_s \frac{8FD}{\pi d^3}$$

$$f_n = \frac{1}{2}\sqrt{\frac{kg}{w}} = \frac{1}{2}\sqrt{\frac{d^4 G}{8D^3 N} \frac{g}{\rho(\pi d^2/4)\pi D N}} = \frac{\sqrt{Gg}\, d}{2\sqrt{2}\rho\pi D^2 N}$$

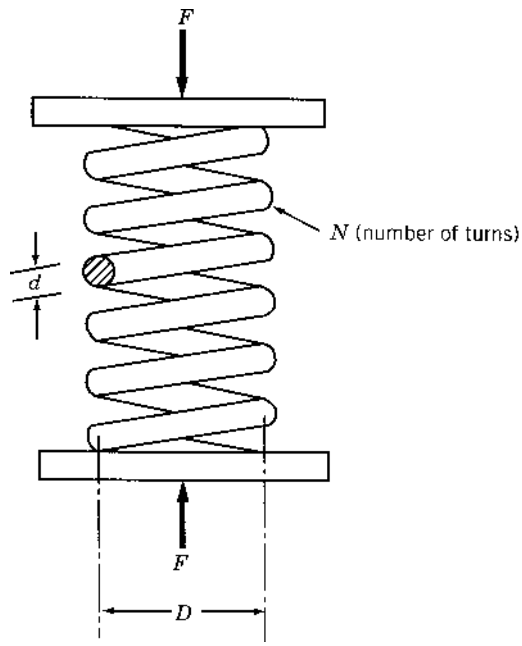

Figure 1.11 Helical spring.

where G is the shear modulus, F the compressive load on the spring, w the weight of the spring, ρ the weight density of the spring, and K_s the shear stress correction factor. Assume that the material is spring steel with $G = 12 \times 10^6$ psi and $\rho = 0.3\,\text{lb/in}^3$, and the shear stress correction factor is $K_s \approx 1.05$.

SOLUTION The design vector is given by

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d \\ D \\ N \end{Bmatrix}$$

and the objective function by

$$f(\mathbf{X}) = \text{weight} = \frac{\pi d^2}{4} \pi D N \rho \qquad (E_1)$$

The constraints can be expressed as

$$\text{deflection} = \frac{F}{k} = \frac{8FD^3N}{d^4G} \leq 0.1$$

that is,

$$g_1(\mathbf{X}) = \frac{d^4G}{80FD^3N} > 1 \qquad (E_2)$$

$$\text{shear stress} = K_s \frac{8FD}{\pi d^3} \leq 10{,}000$$

that is,

$$g_2(\mathbf{X}) = \frac{1250\pi d^3}{K_s F D} > 1 \tag{E$_3$}$$

$$\text{natural frequency} = \frac{\sqrt{Gg}}{2\sqrt{2\rho}\pi} \frac{d}{D^2 N} \geq 100$$

that is,

$$g_3(\mathbf{X}) = \frac{\sqrt{Gg}\, d}{200\sqrt{2\rho}\pi\, D^2 N} > 1 \tag{E$_4$}$$

Since the equality sign is not included (along with the inequality symbol, >) in the constraints of Eqs. (E$_2$) to (E$_4$), the design variables are to be restricted to positive values as

$$d > 0, \quad D > 0, \quad N > 0 \tag{E$_5$}$$

By substituting the known data, F = weight of the milling machine/4 = 1250 lb, $\rho = 0.3$ lb/in^3, $G = 12 \times 10^6$ psi, and $K_s = 1.05$, Eqs. (E$_1$) to (E$_4$) become

$$f(\mathbf{X}) = \tfrac{1}{4}\pi^2(0.3)d^2DN = 0.7402x_1^2x_2x_3 \tag{E$_6$}$$

$$g_1(\mathbf{X}) = \frac{d^4(12 \times 10^6)}{80(1250)D^3N} = 120x_1^4x_2^{-3}x_3^{-1} > 1 \tag{E$_7$}$$

$$g_2(\mathbf{X}) = \frac{1250\pi d^3}{1.05(1250)D} = 2.992x_1^3x_2^{-1} > 1 \tag{E$_8$}$$

$$g_3(\mathbf{X}) = \frac{\sqrt{Gg}\, d}{200\sqrt{2\rho}\pi\, D^2N} = 139.8388x_1x_2^{-2}x_3^{-1} > 1 \tag{E$_9$}$$

It can be seen that the objective function, $f(\mathbf{X})$, and the constraint functions, $g_1(\mathbf{X})$ to $g_3(\mathbf{X})$, are posynomials and hence the problem is a GMP problem.

Quadratic Programming Problem. A quadratic programming problem is a nonlinear programming problem with a quadratic objective function and linear constraints. It is usually formulated as follows:

$$F(\mathbf{X}) = c + \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij} x_i x_j \tag{1.9}$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i = b_j, \quad j = 1, 2, \ldots, m$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, n$$

where c, q_i, Q_{ij}, a_{ij}, and b_j are constants.

Example 1.5 A manufacturing firm produces two products, A and B, using two limited resources. The maximum amounts of resources 1 and 2 available per day are 1000 and 250 units, respectively. The production of 1 unit of product A requires 1 unit of resource 1 and 0.2 unit of resource 2, and the production of 1 unit of product B requires 0.5 unit of resource 1 and 0.5 unit of resource 2. The unit costs of resources 1 and 2 are given by the relations $(0.375 - 0.00005u_1)$ and $(0.75 - 0.0001u_2)$, respectively, where u_i denotes the number of units of resource i used $(i = 1, 2)$. The selling prices per unit of products A and B, p_A and p_B, are given by

$$p_A = 2.00 - 0.0005x_A - 0.00015x_B$$

$$p_B = 3.50 - 0.0002x_A - 0.0015x_B$$

where x_A and x_B indicate, respectively, the number of units of products A and B sold. Formulate the problem of maximizing the profit assuming that the firm can sell all the units it manufactures.

SOLUTION Let the design variables be the number of units of products A and B manufactured per day:

$$\mathbf{X} = \begin{Bmatrix} x_A \\ x_B \end{Bmatrix}$$

The requirement of resource 1 per day is $(x_A + 0.5x_B)$ and that of resource 2 is $(0.2x_A + 0.5x_B)$ and the constraints on the resources are

$$x_A + 0.5x_B \leq 1000 \tag{E$_1$}$$

$$0.2x_A + 0.5x_B \leq 250 \tag{E$_2$}$$

The lower bounds on the design variables can be taken as

$$x_A \geq 0 \tag{E$_3$}$$

$$x_B \geq 0 \tag{E$_4$}$$

The total cost of resources 1 and 2 per day is

$$(x_A + 0.5x_B)[0.375 - 0.00005(x_A + 0.5x_B)] + (0.2x_A + 0.5x_B)[0.750 - 0.0001(0.2x_A + 0.5x_B)]$$

and the return per day from the sale of products A and B is

$$x_A(2.00 - 0.0005x_A - 0.00015x_B) + x_B(3.50 - 0.0002x_A - 0.0015x_B)$$

The total profit is given by the total return minus the total cost. Since the objective function to be minimized is the negative of the profit per day, $f(\mathbf{X})$ is given by

$$\begin{aligned} f(\mathbf{X}) = {} & (x_A + 0.5x_B)[0.375 - 0.00005(x_A + 0.5x_B)] \\ & + (0.2x_A + 0.5x_B)[0.750 - 0.0001(0.2x_A + 0.5x_B)] \\ & - x_A(2.00 - 0.0005x_A - 0.00015x_B) \\ & - x_B(3.50 - 0.0002x_A - 0.0015x_B) \end{aligned} \tag{E$_5$}$$

As the objective function [Eq. (E_5)] is a quadratic and the constraints [Eqs. (E_1) to (E_4)] are linear, the problem is a quadratic programming problem.

Linear Programming Problem. If the objective function and all the constraints in Eq. (1.1) are linear functions of the design variables, the mathematical programming problem is called a *linear programming* (LP) *problem*. A linear programming problem is often stated in the following standard form:

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$\text{which minimizes } f(\mathbf{X}) = \sum_{i=1}^{n} c_i x_i$$

subject to the constraints (1.10)

$$\sum_{i=1}^{n} a_{ij} x_i = b_j, \quad j = 1, 2, \ldots, m$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, n$$

where c_i, a_{ij}, and b_j are constants.

Example 1.6 A scaffolding system consists of three beams and six ropes as shown in Fig. 1.12. Each of the top ropes A and B can carry a load of W_1, each of the middle ropes C and D can carry a load of W_2, and each of the bottom ropes E and F can carry a load of W_3. If the loads acting on beams 1, 2, and 3 are x_1, x_2, and x_3, respectively, as shown in Fig. 1.12, formulate the problem of finding the maximum

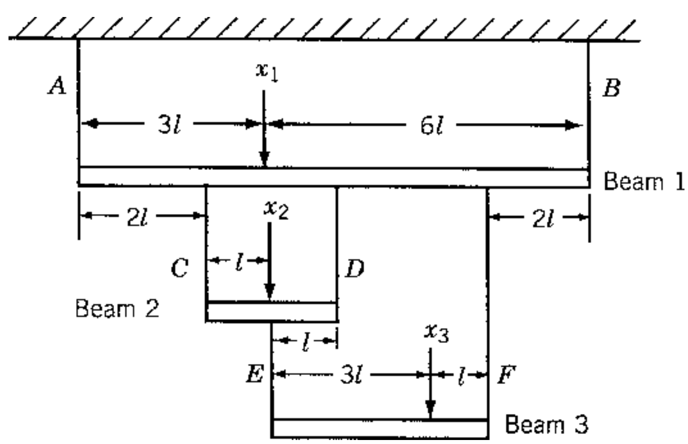

Figure 1.12 Scaffolding system with three beams.

load $(x_1 + x_2 + x_3)$ that can be supported by the system. Assume that the weights of the beams 1, 2, and 3 are w_1, w_2, and w_3, respectively, and the weights of the ropes are negligible.

SOLUTION Assuming that the weights of the beams act through their respective middle points, the equations of equilibrium for vertical forces and moments for each of the three beams can be written as

For beam 3:

$$T_E + T_F = x_3 + w_3$$
$$x_3(3l) + w_3(2l) - T_F(4l) = 0$$

For beam 2:

$$T_C + T_D - T_E = x_2 + w_2$$
$$x_2(l) + w_2(l) + T_E(l) - T_D(2l) = 0$$

For beam 1:

$$T_A + T_B - T_C - T_D - T_F = x_1 + w_1$$
$$x_1(3l) + w_1(\tfrac{9}{2}l) - T_B(9l) + T_C(2l) + T_D(4l) + T_F(7l) = 0$$

where T_i denotes the tension in rope i. The solution of these equations gives

$$T_F = \tfrac{3}{4}x_3 + \tfrac{1}{2}w_3$$
$$T_E = \tfrac{1}{4}x_3 + \tfrac{1}{2}w_3$$
$$T_D = \tfrac{1}{2}x_2 + \tfrac{1}{8}x_3 + \tfrac{1}{2}w_2 + \tfrac{1}{4}w_3$$
$$T_C = \tfrac{1}{2}x_2 + \tfrac{1}{8}x_3 + \tfrac{1}{2}w_2 + \tfrac{1}{4}w_3$$
$$T_B = \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3 + \tfrac{1}{2}w_1 + \tfrac{1}{3}w_2 + \tfrac{5}{9}w_3$$
$$T_A = \tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{1}{3}x_3 + \tfrac{1}{2}w_1 + \tfrac{2}{3}w_2 + \tfrac{4}{9}w_3$$

The optimization problem can be formulated by choosing the design vector as

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Since the objective is to maximize the total load

$$f(\mathbf{X}) = -(x_1 + x_2 + x_3) \tag{E$_1$}$$

The constraints on the forces in the ropes can be stated as

$$T_A \le W_1 \tag{E$_2$}$$
$$T_B \le W_1 \tag{E$_3$}$$
$$T_C \le W_2 \tag{E$_4$}$$

$$T_D \le W_2 \tag{E$_5$}$$

$$T_E \le W_3 \tag{E$_6$}$$

$$T_F \le W_3 \tag{E$_7$}$$

Finally, the nonnegativity requirement of the design variables can be expressed as

$$\begin{aligned} x_1 &\ge 0 \\ x_2 &\ge 0 \\ x_3 &\ge 0 \end{aligned} \tag{E$_8$}$$

Since all the equations of the problem (E_1) to (E_8), are linear functions of x_1, x_2, and x_3, the problem is a linear programming problem.

1.5.5 Classification Based on the Permissible Values of the Design Variables

Depending on the values permitted for the design variables, optimization problems can be classified as integer and real-valued programming problems.

Integer Programming Problem. If some or all of the design variables $x_1, x_2, \ldots, x_n$ of an optimization problem are restricted to take on only integer (or discrete) values, the problem is called an *integer programming problem*. On the other hand, if all the design variables are permitted to take any real value, the optimization problem is called a *real-valued programming problem*. According to this definition, the problems considered in Examples 1.1 to 1.6 are real-valued programming problems.

Example 1.7 A cargo load is to be prepared from five types of articles. The weight w_i, volume v_i, and monetary value c_i of different articles are given below.

Article type	w_i	v_i	c_i
1	4	9	5
2	8	7	6
3	2	4	3
4	5	3	2
5	3	8	8

Find the number of articles x_i selected from the ith type ($i = 1, 2, 3, 4, 5$), so that the total monetary value of the cargo load is a maximum. The total weight and volume of the cargo cannot exceed the limits of 2000 and 2500 units, respectively.

SOLUTION Let x_i be the number of articles of type i ($i = 1$ to 5) selected. Since it is not possible to load a fraction of an article, the variables x_i can take only integer values.

The objective function to be maximized is given by

$$f(\mathbf{X}) = 5x_1 + 6x_2 + 3x_3 + 2x_4 + 8x_5 \tag{E$_1$}$$

and the constraints by

$$4x_1 + 8x_2 + 2x_3 + 5x_4 + 3x_5 \le 2000 \tag{E$_2$}$$

$$9x_1 + 7x_2 + 4x_3 + 3x_4 + 8x_5 \le 2500 \tag{E$_3$}$$

$$x_i \ge 0 \text{ and integral}, \quad i = 1, 2, \ldots, 5 \tag{E$_4$}$$

Since x_i are constrained to be integers, the problem is an integer programming problem.

1.5.6 Classification Based on the Deterministic Nature of the Variables

Based on the deterministic nature of the variables involved, optimization problems can be classified as deterministic and stochastic programming problems.

Stochastic Programming Problem. A stochastic programming problem is an optimization problem in which some or all of the parameters (design variables and/or preassigned parameters) are probabilistic (nondeterministic or stochastic). According to this definition, the problems considered in Examples 1.1 to 1.7 are deterministic programming problems.

Example 1.8 Formulate the problem of designing a minimum-cost rectangular under-reinforced concrete beam that can carry a bending moment M with a probability of at least 0.95. The costs of concrete, steel, and formwork are given by $C_c = \$200/\text{m}^3$, $C_s = \$5000/\text{m}^3$, and $C_f = \$40/\text{m}^2$ of surface area. The bending moment M is a probabilistic quantity and varies between 1×10^5 and 2×10^5 N-m with a uniform probability. The strengths of concrete and steel are also uniformly distributed probabilistic quantities whose lower and upper limits are given by

$$f_c = 25 \text{ and } 35 \text{ MPa}$$

$$f_s = 500 \text{ and } 550 \text{ MPa}$$

Assume that the area of the reinforcing steel and the cross-sectional dimensions of the beam are deterministic quantities.

SOLUTION The breadth b in meters, the depth d in meters, and the area of reinforcing steel A_s in square meters are taken as the design variables x_1, x_2, and x_3, respectively (Fig. 1.13). The cost of the beam per meter length is given by

$$\begin{aligned} f(\mathbf{X}) &= \text{cost of steet} + \text{cost of concrete} + \text{cost of formwork} \\ &= A_s C_s + (bd - A_s)C_c + 2(b + d)C_f \end{aligned} \tag{E$_1$}$$

The resisting moment of the beam section is given by [1.119]

$$M_R = A_s f_s \left(d - 0.59 \frac{A_s f_s}{f_c b} \right)$$

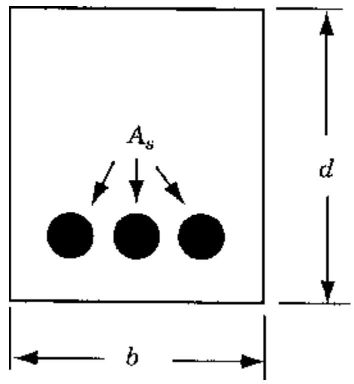

Figure 1.13 Cross section of a reinforced concrete beam.

and the constraint on the bending moment can be expressed as [1.120]

$$P[M_R - M \geq 0] = P\left[A_s f_s \left(d - 0.59\frac{A_s f_s}{f_c b}\right) - M \geq 0\right] \geq 0.95 \qquad (E_2)$$

where $P[\cdots]$ indicates the probability of occurrence of the event $[\cdots]$.

To ensure that the beam remains underreinforced,† the area of steel is bounded by the balanced steel area $A_s^{(b)}$ as

$$A_s \leq A_s^{(b)} \qquad (E_3)$$

where

$$A_s^{(b)} = (0.542)\frac{f_c}{f_s}\, bd\, \frac{600}{600 + f_s}$$

Since the design variables cannot be negative, we have

$$d \geq 0$$

$$b \geq 0$$

$$A_s \geq 0 \qquad (E_4)$$

Since the quantities M, f_c, and f_s are nondeterministic, the problem is a stochastic programming problem.

1.5.7 Classification Based on the Separability of the Functions

Optimization problems can be classified as separable and nonseparable programming problems based on the separability of the objective and constraint functions.

†If steel area is larger than $A_s^{(b)}$, the beam becomes overreinforced and failure occurs all of a sudden due to lack of concrete strength. If the beam is underreinforced, failure occurs due to lack of steel strength and hence it will be gradual.

Separable Programming Problem.

Definition A function $f(\mathbf{X})$ is said to be *separable* if it can be expressed as the sum of n single-variable functions, $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$, that is,

$$f(\mathbf{X}) = \sum_{i=1}^{n} f_i(x_i) \tag{1.11}$$

A separable programming problem is one in which the objective function and the constraints are separable and can be expressed in standard form as

$$\text{Find } \mathbf{X} \text{ which minimizes } f(\mathbf{X}) = \sum_{i=1}^{n} f_i(x_i) \tag{1.12}$$

subject to

$$g_j(\mathbf{X}) = \sum_{i=1}^{n} g_{ij}(x_i) \le b_j, \quad j = 1, 2, \ldots, m$$

where b_j is a constant.

Example 1.9 A retail store stocks and sells three different models of TV sets. The store cannot afford to have an inventory worth more than \$45,000 at any time. The TV sets are ordered in lots. It costs $\$a_j$ for the store whenever a lot of TV model j is ordered. The cost of one TV set of model j is c_j. The demand rate of TV model j is d_j units per year. The rate at which the inventory costs accumulate is known to be proportional to the investment in inventory at any time, with $q_j = 0.5$, denoting the constant of proportionality for TV model j. Each TV set occupies an area of $s_j = 0.40\,\text{m}^2$ and the maximum storage space available is $90\,\text{m}^2$. The data known from the past experience are given below.

	TV model j		
	1	2	3
Ordering cost, a_j (\$)	50	80	100
Unit cost, c_j (\$)	40	120	80
Demand rate, d_j	800	400	1200

Formulate the problem of minimizing the average annual cost of ordering and storing the TV sets.

SOLUTION Let x_j denote the number of TV sets of model j ordered in each lot ($j = 1, 2, 3$). Since the demand rate per year of model j is d_j, the number of times the TV model j needs to be ordered is d_j/x_j. The cost of ordering TV model j per year is thus $a_j d_j/x_j$, $j = 1, 2, 3$. The cost of storing TV sets of model j per year is $q_j c_j x_j/2$ since the average level of inventory at any time during the year is equal to

$c_j x_j/2$. Thus the objective function (cost of ordering plus storing) can be expressed as

$$f(\mathbf{X}) = \left(\frac{a_1 d_1}{x_1} + \frac{q_1 c_1 x_1}{2}\right) + \left(\frac{a_2 d_2}{x_2} + \frac{q_2 c_2 x_2}{2}\right) + \left(\frac{a_3 d_3}{x_3} + \frac{q_3 c_3 x_3}{2}\right) \tag{E$_1$}$$

where the design vector **X** is given by

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{E$_2$}$$

The constraint on the worth of inventory can be stated as

$$c_1 x_1 + c_2 x_2 + c_3 x_3 \le 45{,}000 \tag{E$_3$}$$

The limitation on the storage area is given by

$$s_1 x_1 + s_2 x_2 + s_3 x_3 \le 90 \tag{E$_4$}$$

Since the design variables cannot be negative, we have

$$x_j \ge 0, \quad j = 1, 2, 3 \tag{E$_5$}$$

By substituting the known data, the optimization problem can be stated as follows:

Find **X** which minimizes

$$f(\mathbf{X}) = \left(\frac{40{,}000}{x_1} + 10x_1\right) + \left(\frac{32{,}000}{x_2} + 30x_2\right) + \left(\frac{120{,}000}{x_3} + 20x_3\right) \tag{E$_6$}$$

subject to

$$g_1(\mathbf{X}) = 40x_1 + 120x_2 + 80x_3 \le 45{,}000 \tag{E$_7$}$$

$$g_2(\mathbf{X}) = 0.40(x_1 + x_2 + x_3) \le 90 \tag{E$_8$}$$

$$g_3(\mathbf{X}) = -x_1 \le 0 \tag{E$_9$}$$

$$g_4(\mathbf{X}) = -x_2 \le 0 \tag{E$_{10}$}$$

$$g_5(\mathbf{X}) = -x_3 \le 0 \tag{E$_{11}$}$$

It can be observed that the optimization problem stated in Eqs. (E_6) to (E_{11}) is a separable programming problem.

1.5.8 Classification Based on the Number of Objective Functions

Depending on the number of objective functions to be minimized, optimization problems can be classified as single- and multiobjective programming problems. According to this classification, the problems considered in Examples 1.1 to 1.9 are single objective programming problems.

Multiobjective Programming Problem. A multiobjective programming problem can be stated as follows:

$$\text{Find } \mathbf{X} \text{ which minimizes } f_1(\mathbf{X}), f_2(\mathbf{X}), \ldots, f_k(\mathbf{X})$$

subject to (1.13)

$$g_j(\mathbf{X}) \le 0, \quad j = 1, 2, \ldots, m$$

where $f_1, f_2, \ldots, f_k$ denote the objective functions to be minimized simultaneously.

Example 1.10 A uniform column of rectangular cross section is to be constructed for supporting a water tank of mass M (Fig. 1.14). It is required (1) to minimize the mass of the column for economy, and (2) to maximize the natural frequency of transverse vibration of the system for avoiding possible resonance due to wind. Formulate the problem of designing the column to avoid failure due to direct compression and buckling. Assume the permissible compressive stress to be $\sigma_{\max}$.

SOLUTION Let $x_1 = b$ and $x_2 = d$ denote the cross-sectional dimensions of the column. The mass of the column (m) is given by

$$m = \rho b d l = \rho l x_1 x_2 \tag{E$_1$}$$

where ρ is the density and l is the height of the column. The natural frequency of transverse vibration of the water tank (ω), by treating it as a cantilever beam with a tip mass M, can be obtained as [1.118]

$$\omega = \left[\frac{3EI}{(M + \frac{33}{140} m) l^3} \right]^{1/2} \tag{E$_2$}$$

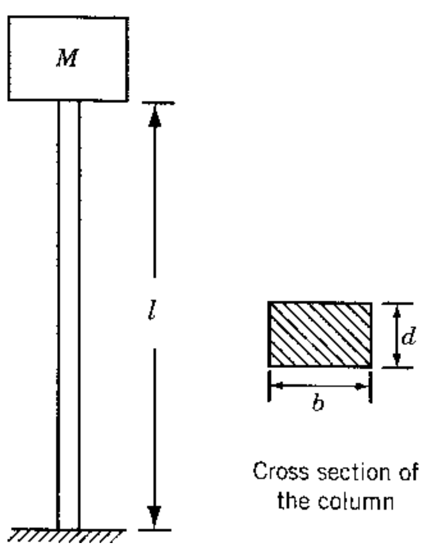

Figure 1.14 Water tank on a column.

where E is the Young's modulus and I is the area moment of inertia of the column given by

$$I = \frac{1}{12} b d^3 \tag{E$_3$}$$

The natural frequency of the water tank can be maximized by minimizing $-\omega$. With the help of Eqs. (E_1) and (E_3), Eq. (E_2) can be rewritten as

$$\omega = \left[\frac{E x_1 x_2^3}{4 l^3 (M + \frac{33}{140} \rho l x_1 x_2)} \right]^{1/2} \tag{E$_4$}$$

The direct compressive stress (σ_c) in the column due to the weight of the water tank is given by

$$\sigma_c = \frac{Mg}{bd} = \frac{Mg}{x_1 x_2} \tag{E$_5$}$$

and the buckling stress for a fixed-free column (σ_b) is given by [1.121]

$$\sigma_b = \left(\frac{\pi^2 E I}{4 l^2} \right) \frac{1}{bd} = \frac{\pi^2 E x_2^2}{48 l^2} \tag{E$_6$}$$

To avoid failure of the column, the direct stress has to be restricted to be less than $\sigma_{\max}$ and the buckling stress has to be constrained to be greater than the direct compressive stress induced.

Finally, the design variables have to be constrained to be positive. Thus the multiobjective optimization problem can be stated as follows:

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \text{ which minimizes}$$

$$f_1(\mathbf{X}) = \rho l x_1 x_2 \tag{E$_7$}$$

$$f_2(\mathbf{X}) = -\left[\frac{E x_1 x_2^3}{4 l^2 (M + \frac{33}{140} \rho l x_1 x_2)} \right]^{1/2} \tag{E$_8$}$$

subject to

$$g_1(\mathbf{X}) = \frac{Mg}{x_1 x_2} - \sigma_{\max} \le 0 \tag{E$_9$}$$

$$g_2(\mathbf{X}) = \frac{Mg}{x_1 x_2} - \frac{\pi^2 E x_2^2}{48 l^2} \le 0 \tag{E$_{10}$}$$

$$g_3(\mathbf{X}) = -x_1 \le 0 \tag{E$_{11}$}$$

$$g_4(\mathbf{X}) = -x_2 \le 0 \tag{E$_{12}$}$$

1.6 OPTIMIZATION TECHNIQUES

The various techniques available for the solution of different types of optimization problems are given under the heading of mathematical programming techniques in Table 1.1. The classical methods of differential calculus can be used to find the unconstrained maxima and minima of a function of several variables. These methods assume that the function is differentiable twice with respect to the design variables and the derivatives are continuous. For problems with equality constraints, the Lagrange multiplier method can be used. If the problem has inequality constraints, the Kuhn–Tucker conditions can be used to identify the optimum point. But these methods lead to a set of nonlinear simultaneous equations that may be difficult to solve. The classical methods of optimization are discussed in Chapter 2.

The techniques of nonlinear, linear, geometric, quadratic, or integer programming can be used for the solution of the particular class of problems indicated by the name of the technique. Most of these methods are numerical techniques wherein an approximate solution is sought by proceeding in an iterative manner by starting from an initial solution. Linear programming techniques are described in Chapters 3 and 4. The quadratic programming technique, as an extension of the linear programming approach, is discussed in Chapter 4. Since nonlinear programming is the most general method of optimization that can be used to solve any optimization problem, it is dealt with in detail in Chapters 5–7. The geometric and integer programming methods are discussed in Chapters 8 and 10, respectively. The dynamic programming technique, presented in Chapter 9, is also a numerical procedure that is useful primarily for the solution of optimal control problems. Stochastic programming deals with the solution of optimization problems in which some of the variables are described by probability distributions. This topic is discussed in Chapter 11.

In Chapter 12 we discuss calculus of variations, optimal control theory, and optimality criteria methods. The modern methods of optimization, including genetic algorithms, simulated annealing, particle swarm optimization, ant colony optimization, neural network-based optimization, and fuzzy optimization, are presented in Chapter 13. Several practical aspects of optimization are outlined in Chapter 14. The reduction of size of optimization problems, fast reanalysis techniques, the efficient computation of the derivatives of static displacements and stresses, eigenvalues and eigenvectors, and transient response are outlined. The aspects of sensitivity of optimum solution to problem parameters, multilevel optimization, parallel processing, and multiobjective optimization are also presented in this chapter.

1.7 ENGINEERING OPTIMIZATION LITERATURE

The literature on engineering optimization is large and diverse. Several text-books are available and dozens of technical periodicals regularly publish papers related to engineering optimization. This is primarily because optimization is applicable to all areas of engineering. Researchers in many fields must be attentive to the developments in the theory and applications of optimization.

The most widely circulated journals that publish papers related to engineering optimization are *Engineering Optimization, ASME Journal of Mechanical Design, AIAA Journal, ASCE Journal of Structural Engineering, Computers and Structures, International Journal for Numerical Methods in Engineering, Structural Optimization, Journal of Optimization Theory and Applications, Computers and Operations Research, Operations Research*, *Management Science, Evolutionary Computation, IEEE Transactions on Evolutionary Computation, European Journal of Operations Research, IEEE Transactions on Systems, Man and Cybernetics*, and *Journal of Heuristics*. Many of these journals are cited in the chapter references.

1.8 SOLUTION OF OPTIMIZATION PROBLEMS USING MATLAB

The solution of most practical optimization problems requires the use of computers. Several commercial software systems are available to solve optimization problems that arise in different engineering areas. MATLAB is a popular software that is used for the solution of a variety of scientific and engineering problems.† MATLAB has several toolboxes each developed for the solution of problems from a specific scientific area. The specific toolbox of interest for solving optimization and related problems is called the *optimization toolbox*. It contains a library of programs or m-files, which can be used for the solution of minimization, equations, least squares curve fitting, and related problems. The basic information necessary for using the various programs can be found in the user's guide for the optimization toolbox [1.124]. The programs or m-files, also called functions, available in the minimization section of the optimization toolbox are given in Table 1.2. The use of the programs listed in Table 1.2 is demonstrated at the end of different chapters of the book. Basically, the solution procedure involves three steps after formulating the optimization problem in the format required by the MATLAB program (or function) to be used. In most cases, this involves stating the objective function for minimization and the constraints in "$\leq$" form with zero or constant value on the righthand side of the inequalities. After this, step 1 involves writing an m-file for the objective function. Step 2 involves writing an m-file for the constraints. Step 3 involves setting the various parameters at proper values depending on the characteristics of the problem and the desired output and creating an appropriate file to invoke the desired MATLAB program (and coupling the m-files created to define the objective and constraints functions of the problem). As an example, the use of the program, `fmincon`, for the solution of a constrained nonlinear programming problem is demonstrated in Example 1.11.

Example 1.11 Find the solution of the following nonlinear optimization problem (same as the problem in Example 1.1) using the MATLAB function `fmincon`:

$$\text{Minimize } f(x_1, x_2) = 9.82x_1x_2 + 2x_1$$

subject to

$$g_1(x_1, x_2) = \frac{2500}{\pi x_1 x_2} - 500 \leq 0$$

†The basic concepts and procedures of MATLAB are summarized in Appendix C.

Table 1.2 MATLAB Programs or Functions for Solving Optimization Problems

Type of optimization problem	Standard form for solution by MATLAB	Name of MATLAB program or function to solve the problem
Function of one variable or scalar minimization	Find x to minimize $f(x)$ with $x_1 < x < x_2$	`fminbnd`
Unconstrained minimization of function of several variables	Find $\mathbf{x}$ to minimize $f(\mathbf{x})$	`fminunc` or `fminsearch`
Linear programming problem	Find $\mathbf{x}$ to minimize $\mathbf{f}^T\mathbf{x}$ subject to $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`linprog`
Quadratic programming problem	Find $\mathbf{x}$ to minimize $\frac{1}{2}\mathbf{x}^T[H]\mathbf{x} + \mathbf{f}^T\mathbf{x}$ subject to $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`quadprog`
Minimization of function of several variables subject to constraints	Find $\mathbf{x}$ to minimize $f(\mathbf{x})$ subject to $\mathbf{c}(\mathbf{x}) \le \mathbf{0}$, $\mathbf{c}_{\text{eq}} = \mathbf{0}$ $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`fmincon`
Goal attainment problem	Find $\mathbf{x}$ and γ to minimize γ such that $F(\mathbf{x}) - \mathbf{w}\gamma \le \mathbf{goal}$, $\mathbf{c}(\mathbf{x}) \le \mathbf{0}$, $\mathbf{c}_{\text{eq}} = \mathbf{0}$ $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`fgoalattain`
Minimax problem	$\underset{\mathbf{x}}{\text{Minimize}}\ \underset{[F_i]}{\text{Max}}\ [F_i(\mathbf{x})]$ such that $\mathbf{c}(\mathbf{x}) \le \mathbf{0}$, $\mathbf{c}_{\text{eq}} = \mathbf{0}$ $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, $\mathbf{l} \le \mathbf{x} \le \mathbf{u}$	`fminimax`
Binary integer programming problem	Find $\mathbf{x}$ to minimize $\mathbf{f}^T\mathbf{x}$ subject to $[A]\mathbf{x} \le \mathbf{b}$, $[A_{\text{eq}}]\mathbf{x} = \mathbf{b}_{\text{eq}}$, each component of $\mathbf{x}$ is binary	`bintprog`

$$g_2(x_1, x_2) = \frac{2500}{\pi x_1 x_2} - \frac{\pi^2(x_1^2 + x_2^2)}{0.5882} \le 0$$

$$g_3(x_1, x_2) = -x_1 + 2 \le 0$$

$$g_4(x_1, x_2) = x_1 - 14 \le 0$$

$$g_5(x_1, x_2) = -x_2 + 0.2 \le 0$$

$$g_6(x_1, x_2) = x_2 - 0.8 \le 0$$

SOLUTION

Step 1: Write an M-file `probofminobj.m` for the objective function.

```
function f= probofminobj (x)
f= 9.82*x(1)*x(2)+2*x(1);
```

Step 2: Write an M-file `conprobformin.m` for the constraints.

```
function [c, ceq] = conprobformin(x)
% Nonlinear inequality constraints
c = [2500/(pi*x(1)*x(2))-500;2500/(pi*x(1)*x(2))-
(pi^2*(x(1)^2+x(2)^2))/0.5882;-x(1)+2;x(1)-14;-x(2)+0.2;
x(2)-0.8];
% Nonlinear equality constraints
ceq = [];
```

Step 3: Invoke constrained optimization program (write this in new matlab file).

```
clc
clear all
warning off
x0 = [7 0.4]; % Starting guess\
fprintf ('The values of function value and constraints
at starting point\n');
f=probofminobj (x0)
[c, ceq] = conprobformin (x0)
options = optimset ('LargeScale', 'off');
[x, fval]=fmincon (@probofminobj, x0, [], [], [], [], [],
[], @conprobformin, options)
fprintf('The values of constraints at optimum solution\n');
[c, ceq] = conprobformin(x) % Check the constraint values at x
```

This produces the solution or output as follows:

```
The values of function value and constraints at starting point
f=
 41.4960
c =
-215.7947
-540.6668
-5.0000
-7.0000
-0.2000
-0.4000
ceq =
  []
Optimization terminated: first-order optimality
measure less
```

```
than options. TolFun and maximum constraint violation
is less
than options.TolCon.
Active inequalities (to within options.TolCon = 1e-006):
lower upper ineqlin ineqnonlin
              1
              2
x=
 5.4510 0.2920
fval =
 26.5310
The values of constraints at optimum solution
c=
 -0.0000
 -0.0000
 -3.4510
 -8.5490
 -0.0920
 -0.5080
ceq =
  []
```

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REVIEW QUESTIONS

1.1 Match the following terms and descriptions:

(a) Free feasible point	$g_j(\mathbf{X}) = 0$
(b) Free infeasible point	Some $g_j(\mathbf{X}) = 0$ and other $g_j(\mathbf{X}) < 0$
(c) Bound feasible point	Some $g_j(\mathbf{X}) = 0$ and other $g_j(\mathbf{X}) \geq 0$
(d) Bound infeasible point	Some $g_j(\mathbf{X}) > 0$ and other $g_j(\mathbf{X}) < 0$
(e) Active constraints	All $g_j(\mathbf{X}) < 0$

1.2 Answer true or false:

(a) Optimization problems are also known as mathematical programming problems.

(b) The number of equality constraints can be larger than the number of design variables.

(c) Preassigned parameters are part of design data in a design optimization problem.

(d) Side constraints are not related to the functionality of the system.

(e) A bound design point can be infeasible.

(f) It is necessary that some $g_j(\mathbf{X}) = 0$ at the optimum point.

(g) An optimal control problem can be solved using dynamic programming techniques.

(h) An integer programming problem is same as a discrete programming problem.

1.3 Define the following terms:

(a) Mathematical programming problem

(b) Trajectory optimization problem

(c) Behavior constraint

(d) Quadratic programming problem

(e) Posynomial

(f) Geometric programming problem

1.4 Match the following types of problems with their descriptions.

(a) Geometric programming problem	Classical optimization problem
(b) Quadratic programming problem	Objective and constraints are quadratic
(c) Dynamic programming problem	Objective is quadratic and constraints are linear
(d) Nonlinear programming problem	Objective and constraints arise from a serial system
(e) Calculus of variations problem	Objective and constraints are polynomials with positive coefficients

1.5 How do you solve a maximization problem as a minimization problem?

1.6 State the linear programming problem in standard form.

1.7 Define an OC problem and give an engineering example.

1.8 What is the difference between linear and nonlinear programming problems?

1.9 What is the difference between design variables and preassigned parameters?

1.10 What is a design space?

1.11 What is the difference between a constraint surface and a composite constraint surface?

1.12 What is the difference between a bound point and a free point in the design space?

1.13 What is a merit function?

1.14 Suggest a simple method of handling multiple objectives in an optimization problem.

1.15 What are objective function contours?

1.16 What is operations research?

1.17 State five engineering applications of optimization.

1.18 What is an integer programming problem?

1.19 What is graphical optimization, and what are its limitations?

1.20 Under what conditions can a polynomial in n variables be called a posynomial?

1.21 Define a stochastic programming problem and give two practical examples.

1.22 What is a separable programming problem?

PROBLEMS

1.1 A fertilizer company purchases nitrates, phosphates, potash, and an inert chalk base at a cost of \$1500, \$500, \$1000, and \$100 per ton, respectively, and produces four fertilizers A, B, C, and D. The production cost, selling price, and composition of the four fertilizers are given below.

			Percentage composition by weight			
Fertilizer	Production cost (\$/ton)	Selling price (\$/ton)	Nitrates	Phosphates	Potash	Inert chalk base
A	100	350	5	10	5	80
B	150	550	5	15	10	70
C	200	450	10	20	10	60
D	250	700	15	5	15	65

During any week, no more than 1000 tons of nitrate, 2000 tons of phosphates, and 1500 tons of potash will be available. The company is required to supply a minimum of 5000 tons of fertilizer A and 4000 tons of fertilizer D per week to its customers; but it is otherwise free to produce the fertilizers in any quantities it pleases. Formulate the problem of finding the quantity of each fertilizer to be produced by the company to maximize its profit.

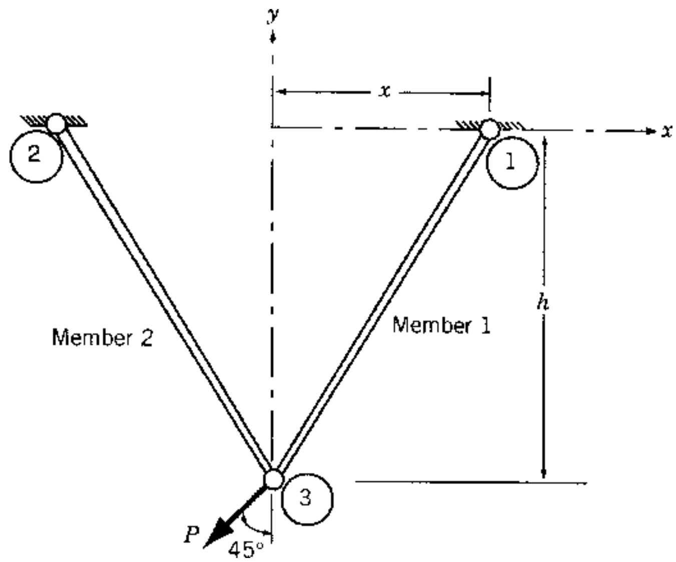

Figure 1.15 Two-bar truss.

1.2 The two-bar truss shown in Fig. 1.15 is symmetric about the y axis. The nondimensional area of cross section of the members A/A_{ref}, and the nondimensional position of joints 1 and 2, x/h, are treated as the design variables x_1 and x_2, respectively, where A_{ref} is the reference value of the area (A) and h is the height of the truss. The coordinates of joint 3 are held constant. The weight of the truss (f_1) and the total displacement of joint 3 under the given load (f_2) are to be minimized without exceeding the permissible stress, σ_0. The weight of the truss and the displacement of joint 3 can be expressed as

$$f_1(\mathbf{X}) = 2\rho h x_2 \sqrt{1 + x_1^2}\, A_{\text{ref}}$$

$$f_2(\mathbf{X}) = \frac{Ph(1 + x_1^2)^{1.5}\sqrt{1 + x_1^4}}{2\sqrt{2}\, E x_1^2 x_2 A_{\text{ref}}}$$

where ρ is the weight density, P the applied load, and E the Young's modulus. The stresses induced in members 1 and 2 (σ_1 and σ_2) are given by

$$\sigma_1(\mathbf{X}) = \frac{P(1 + x_1)\sqrt{(1 + x_1^2)}}{2\sqrt{2}\, x_1 x_2 A_{\text{ref}}}$$

$$\sigma_2(\mathbf{X}) = \frac{P(x_1 - 1)\sqrt{(1 + x_1^2)}}{2\sqrt{2}\, x_1 x_2 A_{\text{ref}}}$$

In addition, upper and lower bounds are placed on design variables x_1 and x_2 as

$$x_i^{\min} \le x_i \le x_i^{\max}; \quad i = 1, 2$$

Find the solution of the problem using a graphical method with (**a**) f_1 as the objective, (**b**) f_2 as the objective, and (**c**) $(f_1 + f_2)$ as the objective for the following data: $E = 30 \times 10^6$ psi,

$\rho = 0.283$ lb/in^3, $P = 10{,}000$ lb, $\sigma_0 = 20{,}000$ psi, $h = 100$ in., $A_{\text{ref}} = 1$ in^2, $x_1^{\min} = 0.1$, $x_2^{\min} = 0.1$, $x_1^{\max} = 2.0$, and $x_2^{\max} = 2.5$.

1.3 Ten jobs are to be performed in an automobile assembly line as noted in the following table:

Job Number	Time required to complete the job (min)	Jobs that must be completed before starting this job
1	4	None
2	8	None
3	7	None
4	6	None
5	3	1, 3
6	5	2, 3, 4
7	1	5, 6
8	9	6
9	2	7, 8
10	8	9

It is required to set up a suitable number of workstations, with one worker assigned to each workstation, to perform certain jobs. Formulate the problem of determining the number of workstations and the particular jobs to be assigned to each workstation to minimize the idle time of the workers as an integer programming problem. *Hint:* Define variables x_{ij} such that $x_{ij} = 1$ if job i is assigned to station j, and $x_{ij} = 0$ otherwise.

1.4 A railroad track of length L is to be constructed over an uneven terrain by adding or removing dirt (Fig. 1.16). The absolute value of the slope of the track is to be restricted to a value of r_1 to avoid steep slopes. The absolute value of the rate of change of the slope is to be limited to a value r_2 to avoid rapid accelerations and decelerations. The absolute value of the second derivative of the slope is to be limited to a value of r_3

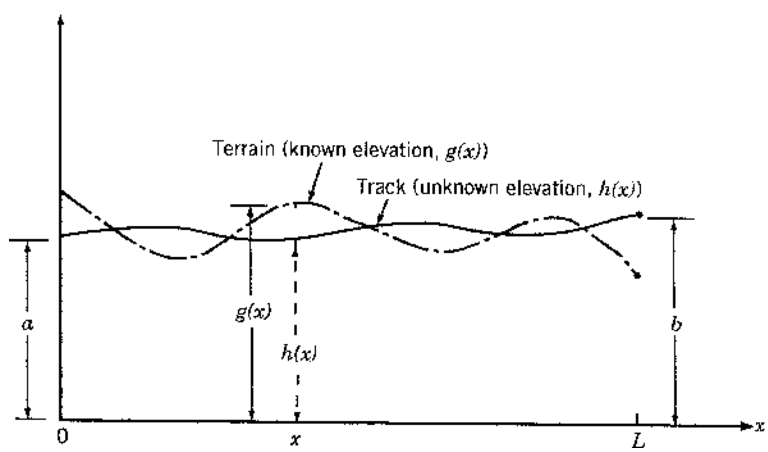

Figure 1.16 Railroad track on an uneven terrain.

to avoid severe jerks. Formulate the problem of finding the elevation of the track to minimize the construction costs as an OC problem. Assume the construction costs to be proportional to the amount of dirt added or removed. The elevation of the track is equal to a and b at $x = 0$ and $x = L$, respectively.

1.5 A manufacturer of a particular product produces x_1 units in the first week and x_2 units in the second week. The number of units produced in the first and second weeks must be at least 200 and 400, respectively, to be able to supply the regular customers. The initial inventory is zero and the manufacturer ceases to produce the product at the end of the second week. The production cost of a unit, in dollars, is given by $4x_i^2$, where x_i is the number of units produced in week $i (i = 1, 2)$. In addition to the production cost, there is an inventory cost of \$10 per unit for each unit produced in the first week that is not sold by the end of the first week. Formulate the problem of minimizing the total cost and find its solution using a graphical optimization method.

1.6 Consider the slider-crank mechanism shown in Fig. 1.17 with the crank rotating at a constant angular velocity ω. Use a graphical procedure to find the lengths of the crank and the connecting rod to maximize the velocity of the slider at a crank angle of $\theta = 30^\circ$ for $\omega = 100$ rad/s. The mechanism has to satisfy Groshof's criterion $l \geq 2.5r$ to ensure 360° rotation of the crank. Additional constraints on the mechanism are given by $0.5 \leq r \leq 10, 2.5 \leq l \leq 25$, and $10 \leq x \leq 20$.

1.7 Solve Problem 1.6 to maximize the acceleration (instead of the velocity) of the slider at $\theta = 30^\circ$ for $\omega = 100$ rad/s.

1.8 It is required to stamp four circular disks of radii R_1, R_2, R_3, and R_4 from a rectangular plate in a fabrication shop (Fig. 1.18). Formulate the problem as an optimization problem to minimize the scrap. Identify the design variables, objective function, and the constraints.

1.9 The torque transmitted (T) by a cone clutch, shown in Fig. 1.19, under uniform pressure condition is given by

$$T = \frac{2\pi f p}{3 \sin \alpha}(R_1^3 - R_2^3)$$

where p is the pressure between the cone and the cup, f the coefficient of friction, α the cone angle, R_1 the outer radius, and R_2 the inner radius.

(a) Find R_1 and R_2 that minimize the volume of the cone clutch with $\alpha = 30^\circ$, $F = 30$ lb, and $f = 0.5$ under the constraints $T \geq 100$ lb-in., $R_1 \geq 2R_2$, $0 \leq R_1 \leq 15$ in., and $0 \leq R_2 \leq 10$ in.

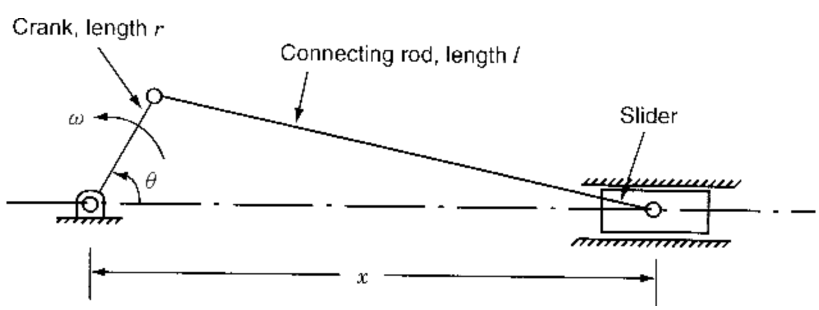

Figure 1.17 Slider-crank mechanism.

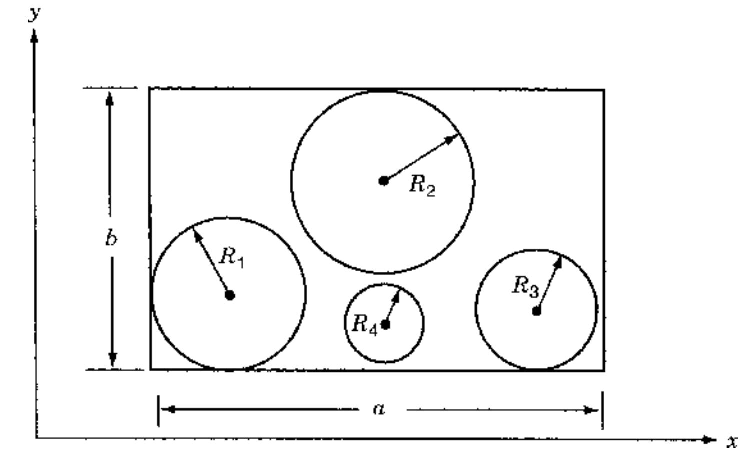

Figure 1.18 Locations of circular disks in a rectangular plate.

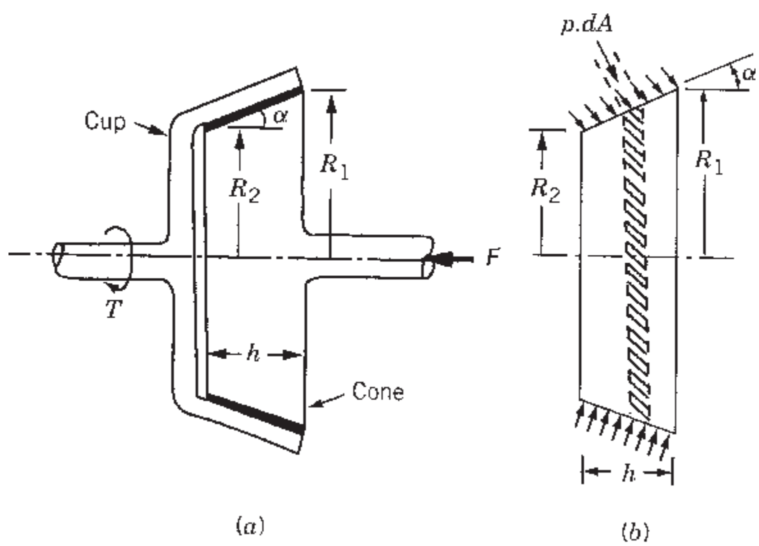

Figure 1.19 Cone clutch.

(b) What is the solution if the constraint $R_1 \geq 2R_2$ is changed to $R_1 \leq 2R_2$?

(c) Find the solution of the problem stated in part (a) by assuming a uniform wear condition between the cup and the cone. The torque transmitted (T) under uniform wear condition is given by

$$T = \frac{\pi f p R_2}{\sin \alpha}(R_1^2 - R_2^2)$$

Note: Use graphical optimization for the solutions.

1.10 A hollow circular shaft is to be designed for minimum weight to achieve a minimum reliability of 0.99 when subjected to a random torque of $(\overline{T}, \sigma_T) = (10^6, 10^4)$ lb-in., where $\overline{T}$ is the mean torque and σ_T is the standard deviation of the torque, T. The permissible shear stress, τ_0, of the material is given by $(\overline{\tau}_0, \sigma_{\tau 0}) = (50{,}000, 5000)$ psi, where $\overline{\tau}_0$ is the mean value and $\sigma_{\tau 0}$ is the standard deviation of τ_0. The maximum induced stress (τ) in the shaft is given by

$$\tau = \frac{T r_o}{J}$$

where r_o is the outer radius and J is the polar moment of inertia of the cross section of the shaft. The manufacturing tolerances on the inner and outer radii of the shaft are specified as ± 0.06 in. The length of the shaft is given by 50 ± 1 in. and the specific weight of the material by $0.3 \pm\ 0.03$ lb/in^3. Formulate the optimization problem and solve it using a graphical procedure. Assume normal distribution for all the random variables and 3σ values for the specified tolerances. *Hints:* (1) The minimum reliability requirement of 0.99 can be expressed, equivalently, as [1.120]

$$z_1 = 2.326 \le \frac{\overline{\tau} - \overline{\tau}_0}{\sqrt{\sigma_\tau^2 + \sigma_{\tau 0}^2}}$$

(2) If $f(x_1, x_2, \ldots, x_n)$ is a function of the random variables $x_1, x_2, \ldots, x_n$, the mean value of $f(\overline{f})$ and the standard deviation of $f(\sigma_f)$ are given by

$$\overline{f} = f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$$

$$\sigma_f = \left[\sum_{i=1}^{n} \left(\left. \frac{\partial f}{\partial x_i} \right|_{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n} \right)^2 \sigma_{xi}^2 \right]^{1/2}$$

where $\overline{x}_i$ is the mean value of x_i, and σ_{xi} is the standard deviation of x_i.

1.11 Certain nonseparable optimization problems can be reduced to a separable form by using suitable transformation of variables. For example, the product term $f = x_1 x_2$ can be reduced to the separable form $f = y_1^2 - y_2^2$ by introducing the transformations

$$y_1 = \tfrac{1}{2}(x_1 + x_2), \quad y_2 = \tfrac{1}{2}(x_1 - x_2)$$

Suggest suitable transformations to reduce the following terms to separable form:

(a) $f = x_1^2 x_2^3,\ x_1 > 0, x_2 > 0$

(b) $f = x_1^{x2},\ x_1 > 0$

1.12 In the design of a shell-and-tube heat exchanger (Fig. 1.20), it is decided to have the total length of tubes equal to at least α_1 [1.10]. The cost of the tube is α_2 per unit length and the cost of the shell is given by $\alpha_3 D^{2.5} L$, where D is the diameter and L is the length of the heat exchanger shell. The floor space occupied by the heat exchanger costs α_4 per unit area and the cost of pumping cold fluid is $\alpha_5 L/d^5 N^2$ per day, where d is the diameter of the tube and N is the number of tubes. The maintenance cost is given by $\alpha_6 NdL$. The thermal energy transferred to the cold fluid is given by $\alpha_7/N^{1.2} dL^{1.4} + \alpha_8/d^{0.2} L$. Formulate the mathematical programming problem of minimizing the overall cost of the heat exchanger with the constraint that the thermal energy transferred be greater than a specified amount α_9. The expected life of the heat exchanger is α_{10} years. Assume that $\alpha_i, i = 1, 2, \ldots, 10$, are known constants, and each tube occupies a cross-sectional square of width and depth equal to d.

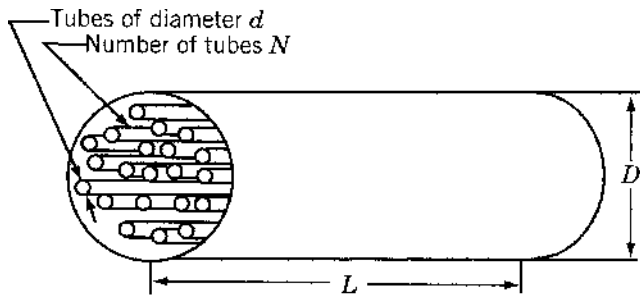

Figure 1.20 Shell-and-tube heat exchanger.

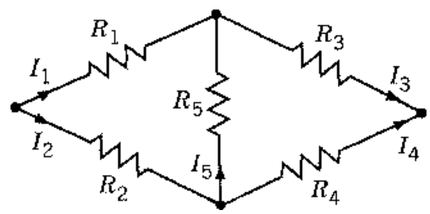

Figure 1.21 Electrical bridge network.

1.13 The bridge network shown in Fig. 1.21 consists of five resistors $R_i (i = 1, 2, \ldots, 5)$. If I_i is the current flowing through the resistance R_i, the problem is to find the resistances $R_1, R_2, \ldots, R_5$ so that the total power dissipated by the network is a minimum. The current I_i can vary between the lower and upper limits $I_{i,\min}$ and $I_{i,\max}$, and the voltage drop, $V_i = R_i I_i$, must be equal to a constant c_i for $1 \le i \le 5$. Formulate the problem as a mathematical programming problem.

1.14 A traveling saleswoman has to cover n towns. She plans to start from a particular town numbered 1, visit each of the other $n - 1$ towns, and return to the town 1. The distance between towns i and j is given by d_{ij}. Formulate the problem of selecting the sequence in which the towns are to be visited to minimize the total distance traveled.

1.15 A farmer has a choice of planting barley, oats, rice, or wheat on his 200-acre farm. The labor, water, and fertilizer requirements, yields per acre, and selling prices are given in the following table:

Type of crop	Labor cost ($)	Water required (m^3)	Fertilizer required (lb)	Yield (lb)	Selling price ($/lb)
Barley	300	10,000	100	1,500	0.5
Oats	200	7,000	120	3,000	0.2
Rice	250	6,000	160	2,500	0.3
Wheat	360	8,000	200	2,000	0.4

The farmer can also give part or all of the land for lease, in which case he gets $200 per acre. The cost of water is $0.02/m^3 and the cost of the fertilizer is $2/lb. Assume that the farmer has no money to start with and can get a maximum loan of $50,000 from the land mortgage bank at an interest of 8 %. He can repay the loan after six months. The irrigation canal cannot supply more than 4×10^5 m^3 of water. Formulate the problem of finding the planting schedule for maximizing the expected returns of the farmer.

1.16 There are two different sites, each with four possible targets (or depths) to drill an oil well. The preparation cost for each site and the cost of drilling at site i to target j are given below:

	Drilling cost to target j				
Site i	1	2	3	4	Preparation cost
1	4	1	9	7	11
2	7	9	5	2	13

Formulate the problem of determining the best site for each target so that the total cost is minimized.

1.17 A four-pole dc motor, whose cross section is shown in Fig. 1.22, is to be designed with the length of the stator and rotor x_1, the overall diameter of the motor x_2, the unnotched radius x_3, the depth of the notches x_4, and the ampere turns x_5 as design variables.

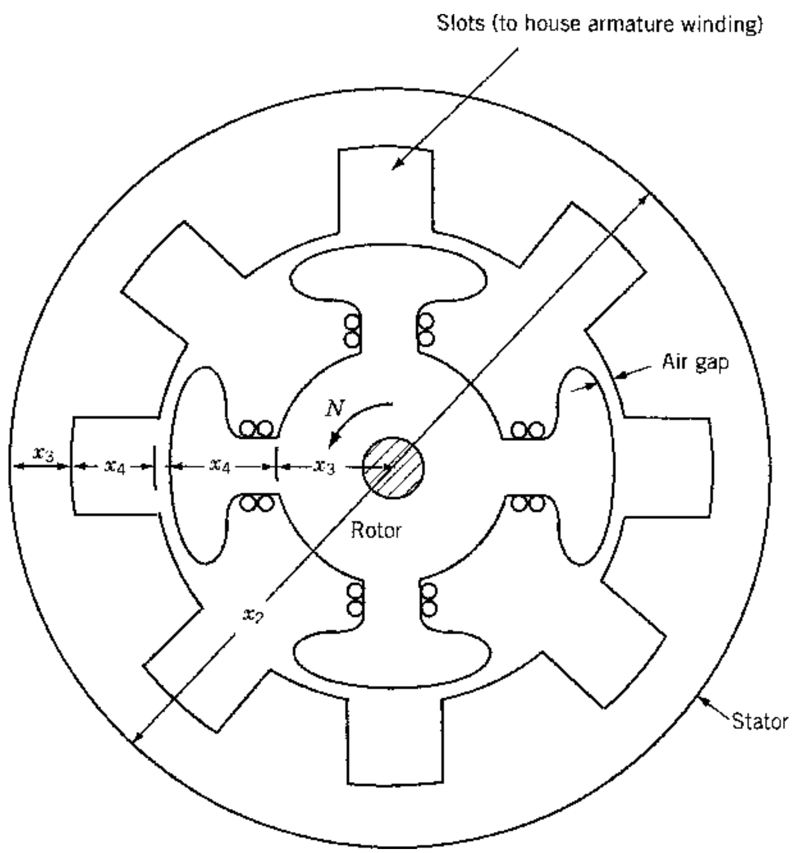

Figure 1.22 Cross section of an idealized motor.

The air gap is to be less than $k_1\sqrt{x_2 + 7.5}$ where k_1 is a constant. The temperature of the external surface of the motor cannot exceed ΔT above the ambient temperature. Assuming that the heat can be dissipated only by radiation, formulate the problem for maximizing the power of the motor [1.59]. *Hints:*

1. The heat generated due to current flow is given by $k_2 x_1 x_2^{-1} x_4^{-1} x_5^2$, where k_2 is a constant. The heat radiated from the external surface for a temperature difference of ΔT is given by $k_3 x_1 x_2 \Delta T$, where k_3 is a constant.
2. The expression for power is given by $k_4 N B x_1 x_3 x_5$, where k_4 is a constant, N is the rotational speed of the rotor, and B is the average flux density in the air gap.
3. The units of the various quantities are as follows. Lengths: centimeter, heat generated, heat dissipated; power: watt; temperature: °C; rotational speed: rpm; flux density: gauss.

1.18 A gas pipeline is to be laid between two cities A and E, making it pass through one of the four locations in each of the intermediate towns B, C, and D (Fig. 1.23). The associated costs are indicated in the following tables.

Costs for A to B and D to E

	Station i			
	1	2	3	4
From A to point i of B	30	35	25	40
From point i of D to E	50	40	35	25

Costs for B to C and C to D

	To:			
From:	1	2	3	4
1	22	18	24	18
2	35	25	15	21
3	24	20	26	20
4	22	21	23	22

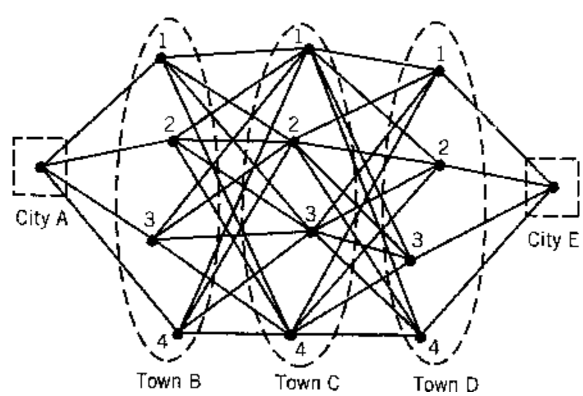

Figure 1.23 Possible paths of the pipeline between A and E.

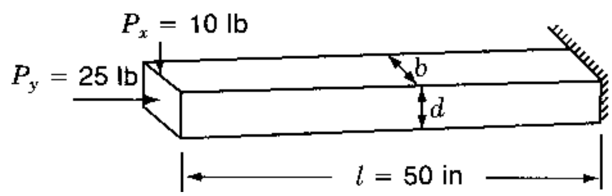

Figure 1.24 Beam-column.

Formulate the problem of minimizing the cost of the pipeline.

1.19 A beam-column of rectangular cross section is required to carry an axial load of 25 lb and a transverse load of 10 lb, as shown in Fig. 1.24. It is to be designed to avoid the possibility of yielding and buckling and for minimum weight. Formulate the optimization problem by assuming that the beam-column can bend only in the vertical (xy) plane. Assume the material to be steel with a specific weight of 0.3 lb/in^3, Young's modulus of 30×10^6 psi, and a yield stress of 30,000 psi. The width of the beam is required to be at least 0.5 in. and not greater than twice the depth. Also, find the solution of the problem graphically. *Hint:* The compressive stress in the beam-column due to P_y is P_y/bd and that due to P_x is

$$\frac{P_x l d}{2I_{zz}} = \frac{6P_x l}{bd^2}$$

The axial buckling load is given by

$$(P_y)_{\text{cri}} = \frac{\pi^2 E I_{zz}}{4l^2} = \frac{\pi^2 E b d^3}{48 l^2}$$

1.20 A two-bar truss is to be designed to carry a load of $2W$ as shown in Fig. 1.25. Both bars have a tubular section with mean diameter d and wall thickness t. The material of the bars has Young's modulus E and yield stress σ_y. The design problem involves the determination of the values of d and t so that the weight of the truss is a minimum and neither yielding nor buckling occurs in any of the bars. Formulate the problem as a nonlinear programming problem.

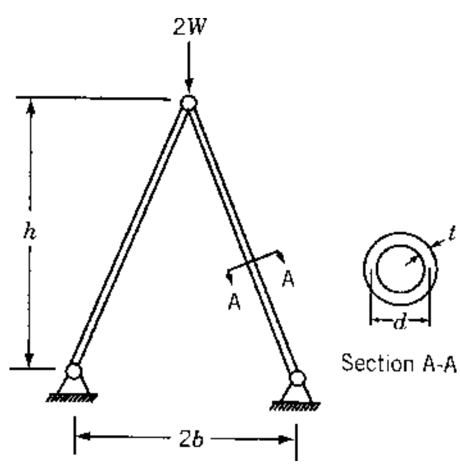

Figure 1.25 Two-bar truss.

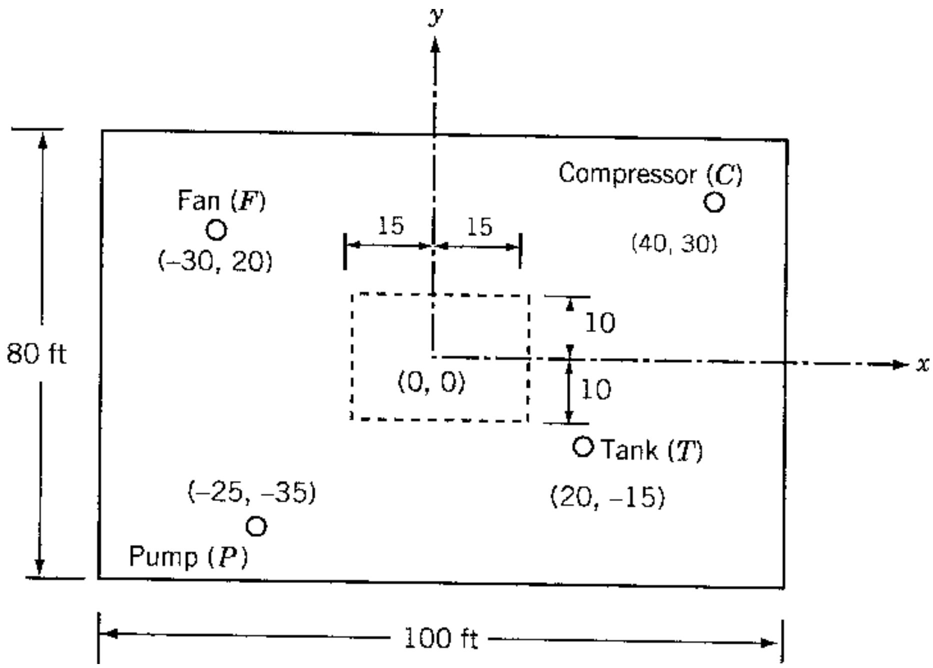

Figure 1.26 Processing plant layout (coordinates in ft).

1.21 Consider the problem of determining the economic lot sizes for four different items. Assume that the demand occurs at a constant rate over time. The stock for the ith item is replenished instantaneously upon request in lots of sizes Q_i. The total storage space available is A, whereas each unit of item i occupies an area d_i. The objective is to find the values of Q_i that optimize the per unit cost of holding the inventory and of ordering subject to the storage area constraint. The cost function is given by

$$C = \sum_{i=1}^{4} \left(\frac{a_i}{Q_i} + b_i Q_i \right), \quad Q_i > 0$$

where a_i and b_i are fixed constants. Formulate the problem as a dynamic programming (optimal control) model. Assume that Q_i is discrete.

1.22 The layout of a processing plant, consisting of a pump (P), a water tank (T), a compressor (C), and a fan (F), is shown in Fig. 1.26. The locations of the various units, in terms of their (x, y) coordinates, are also indicated in this figure. It is decided to add a new unit, a heat exchanger (H), to the plant. To avoid congestion, it is decided to locate H within a rectangular area defined by $\{-15 \le x \le 15, -10 \le y \le 10\}$. Formulate the problem of finding the location of H to minimize the sum of its x and y distances from the existing units, P, T, C, and F.

1.23 Two copper-based alloys (brasses), A and B, are mixed to produce a new alloy, C. The composition of alloys A and B and the requirements of alloy C are given in the following table:

Alloy	Composition by weight			
	Copper	Zinc	Lead	Tin
A	80	10	6	4
B	60	20	18	2
C	≥ 75	≥ 15	≥ 16	≥ 3

If alloy *B* costs twice as much as alloy *A*, formulate the problem of determining the amounts of *A* and *B* to be mixed to produce alloy *C* at a minimum cost.

1.24 An oil refinery produces four grades of motor oil in three process plants. The refinery incurs a penalty for not meeting the demand of any particular grade of motor oil. The capacities of the plants, the production costs, the demands of the various grades of motor oil, and the penalties are given in the following table:

Process plant	Capacity of the plant (kgal/day)	Production cost ($/day) to manufacture motor oil of grade:			
		1	2	3	4
1	100	750	900	1000	1200
2	150	800	950	1100	1400
3	200	900	1000	1200	1600
Demand (kgal/day)		50	150	100	75
Penalty (per each kilogallon shortage)		$10	$12	$16	$20

Formulate the problem of minimizing the overall cost as an LP problem.

1.25 A part-time graduate student in engineering is enrolled in a four-unit mathematics course and a three-unit design course. Since the student has to work for 20 hours a week at a local software company, he can spend a maximum of 40 hours a week to study outside the class. It is known from students who took the courses previously that the numerical grade (g) in each course is related to the study time spent outside the class as $g_m = t_m/6$ and $g_d = t_d/5$, where g indicates the numerical grade ($g = 4$ for A, 3 for B, 2 for C, 1 for D, and 0 for F), t represents the time spent in hours per week to study outside the class, and the subscripts m and d denote the courses, mathematics and design, respectively. The student enjoys design more than mathematics and hence would like to spend at least 75 minutes to study for design for every 60 minutes he spends to study mathematics. Also, as far as possible, the student does not want to spend more time on any course beyond the time required to earn a grade of A. The student wishes to maximize his grade point P, given by $P = 4g_m + 3g_d$, by suitably distributing his study time. Formulate the problem as an LP problem.

1.26 The scaffolding system, shown in Fig. 1.27, is used to carry a load of 10,000 lb. Assuming that the weights of the beams and the ropes are negligible, formulate the problem of determining the values of x_1, x_2, x_3, and x_4 to minimize the tension in ropes A and B while maintaining positive tensions in ropes C, D, E, and F.

1.27 Formulate the problem of minimum weight design of a power screw subjected to an axial load, F, as shown in Fig. 1.28 using the pitch (p), major diameter (d), nut height

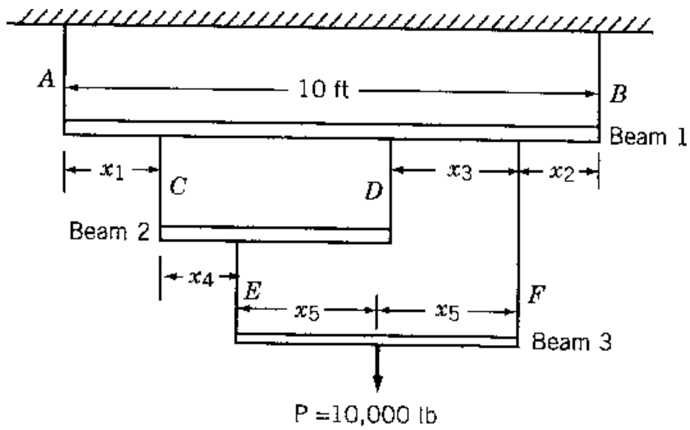

Figure 1.27 Scaffolding system.

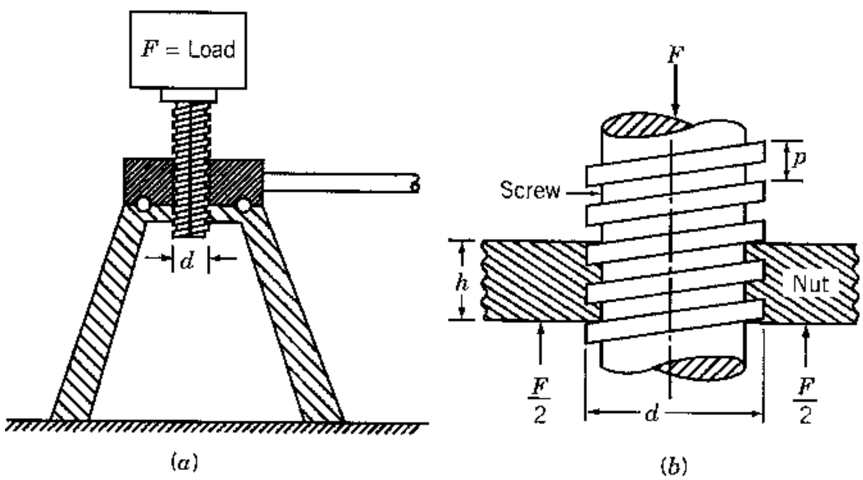

Figure 1.28 Power screw.

(h), and screw length (s) as design variables. Consider the following constraints in the formulation:

1. The screw should be self-locking [1.117].
2. The shear stress in the screw should not exceed the yield strength of the material in shear. Assume the shear strength in shear (according to distortion energy theory), to be $0.577\sigma_y$, where σ_y is the yield strength of the material.
3. The bearing stress in the threads should not exceed the yield strength of the material, σ_y.
4. The critical buckling load of the screw should be less than the applied load, F.

1.28 **(a)** A simply supported beam of hollow rectangular section is to be designed for minimum weight to carry a vertical load F_y and an axial load P as shown in Fig. 1.29. The deflection of the beam in the y direction under the self-weight and F_y should

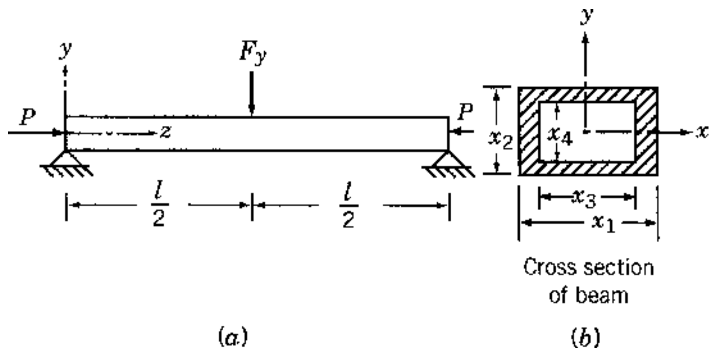

Figure 1.29 Simply supported beam under loads.

not exceed 0.5 in. The beam should not buckle either in the yz or the xz plane under the axial load. Assuming the ends of the beam to be pin ended, formulate the optimization problem using $x_i, i = 1, 2, 3, 4$ as design variables for the following data: $F_y = 300$ lb, $P = 40{,}000$ lb, $l = 120$ in., $E = 30 \times 10^6$ psi, $\rho = 0.284$ lb/in^3, lower bound on x_1 and $x_2 = 0.125$ in, upper bound on x_1, and $x_2 = 4$ in.

(b) Formulate the problem stated in part (a) using x_1 and x_2 as design variables, assuming the beam to have a solid rectangular cross section. Also find the solution of the problem using a graphical technique.

1.29 A cylindrical pressure vessel with hemispherical ends (Fig. 1.30) is required to hold at least 20,000 gallons of a fluid under a pressure of 2500 psia. The thicknesses of the cylindrical and hemispherical parts of the shell should be equal to at least those recommended by section VIII of the ASME pressure vessel code, which are given by

$$t_c = \frac{pR}{Se + 0.4p}$$

$$t_h = \frac{pR}{Se + 0.8p}$$

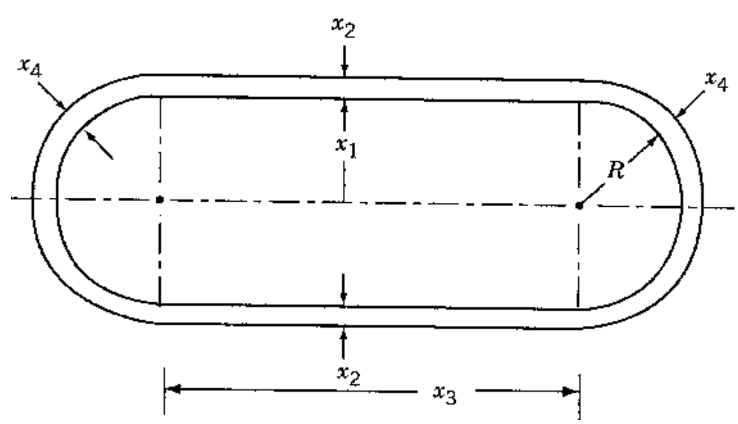

Figure 1.30 Pressure vessel.

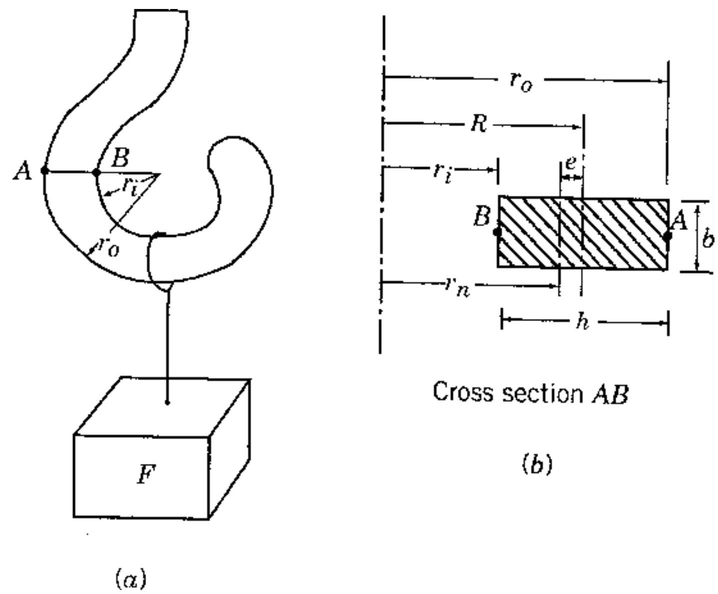

Figure 1.31 Crane hook carrying a load.

where S is the yield strength, e the joint efficiency, p the pressure, and R the radius. Formulate the design problem for minimum structural volume using x_i, $i = 1, 2, 3, 4$, as design variables. Assume the following data: $S = 30{,}000$ psi and $e = 1.0$.

1.30 A crane hook is to be designed to carry a load F as shown in Fig. 1.31. The hook can be modeled as a three-quarter circular ring with a rectangular cross section. The stresses induced at the inner and outer fibers at section AB should not exceed the yield strength of the material. Formulate the problem of minimum volume design of the hook using r_o, r_i, b, and h as design variables. *Note:* The stresses induced at points A and B are given by [1.117]

$$\sigma_A = \frac{Mc_o}{Aer_o}$$

$$\sigma_B = \frac{Mc_i}{Aer_i}$$

where M is the bending moment due to the load ($= FR$), R the radius of the centroid, r_o the radius of the outer fiber, r_i the radius of the inner fiber, c_o the distance of the outer fiber from the neutral axis $= R_o - r_n$, c_i the distance of inner fiber from neutral axis $= r_n - r_i$, r_n the radius of neutral axis, given by

$$r_n = \frac{h}{\text{In}(r_o/r_i)}$$

A the cross-sectional area of the hook $= bh$, and e the distance between the centroidal and neutral axes $= R - r_n$.

1.31 Consider the four-bar truss shown in Fig. 1.32, in which members 1, 2, and 3 have the same cross-sectional area x_1 and the same length l, while member 4 has an area of

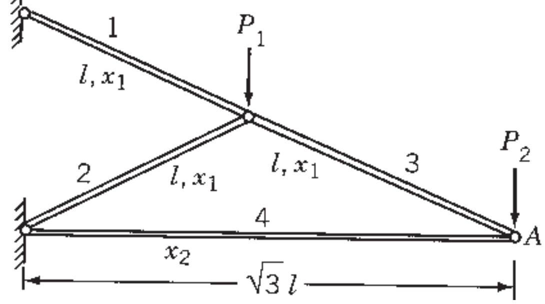

Figure 1.32 Four-bar truss.

cross section x_2 and length $\sqrt{3}\,l$. The truss is made of a lightweight material for which Young's modulus and the weight density are given by 30 × 10^6 psi and 0.03333 lb/in^3, respectively. The truss is subject to the loads $P_1 = 10{,}000$ lb and $P_2 = 20{,}000$ lb. The weight of the truss per unit value of l can be expressed as

$$f = 3x_1(1)(0.03333) + x_2\sqrt{3}(0.03333) = 0.1x_1 + 0.05773x_2$$

The vertical deflection of joint A can be expressed as

$$\delta_A = \frac{0.6}{x_1} + \frac{0.3464}{x_2}$$

and the stresses in members 1 and 4 can be written as

$$\sigma_1 = \frac{5(10{,}000)}{x_1} = \frac{50{,}000}{x_1}, \quad \sigma_4 = \frac{-2\sqrt{3}(10{,}000)}{x_2} = -\frac{34{,}640}{x_2}$$

The weight of the truss is to be minimized with constraints on the vertical deflection of the joint A and the stresses in members 1 and 4. The maximum permissible deflection of joint A is 0.1 in. and the permissible stresses in members are $\sigma_{\max} = 8333.3333$ psi (tension) and $\sigma_{\min} = -4948.5714$ psi (compression). The optimization problem can be stated as a separable programming problem as follows:

$$\text{Minimize } f(x_1, x_2) = 0.1x_1 + 0.05773x_2$$

subject to

$$\frac{0.6}{x_1} + \frac{0.3464}{x_2} - 0.1 \le 0, \quad 6 - x_1 \le 0, \quad 7 - x_2 \le 0$$

Determine the solution of the problem using a graphical procedure.

1.32 A simply supported beam, with a uniform rectangular cross section, is subjected to both distributed and concentrated loads as shown in Fig. 1.33. It is desired to find the cross section of the beam to minimize the weight of the beam while ensuring that the maximum stress induced in the beam does not exceed the permissible stress (σ_0) of the material and the maximum deflection of the beam does not exceed a specified limit (δ_0).
The data of the problem are $P = 10^5$ N, $p_0 = 10^6$ N/m, $L = 1$ m, $E = 207$ GPa, weight density (ρ_w) = 76.5 kN/m^3, $\sigma_0 = 220$ MPa, and $\delta_0 = 0.02$ m.

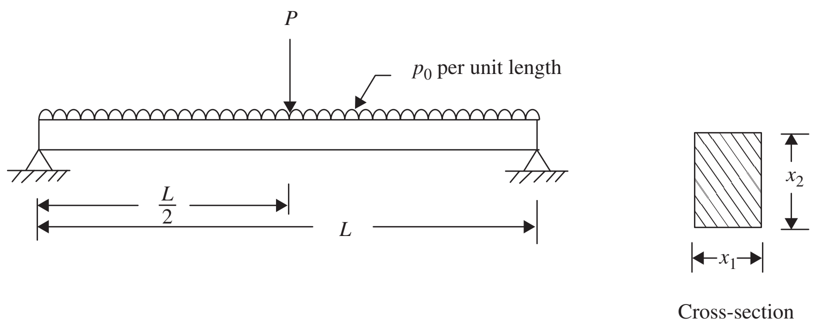

Figure 1.33 A simply supported beam subjected to concentrated and distributed loads.

(a) Formulate the problem as a mathematical programming problem assuming that the cross-sectional dimensions of the beam are restricted as $x_1 \leq x_2$, $0.04m \leq x_1 \leq 0.12m$, and $0.06m \leq x_2 \leq 0.20\ m$.

(b) Find the solution of the problem formulated in part (a) using MATLAB.

(c) Find the solution of the problem formulated in part (a) graphically.

1.33 Solve Problem 1.32, parts (a), (b), and (c), assuming the cross section of the beam to be hollow circular with inner diameter x_1 and outer diameter x_2. Assume the data and bounds on the design variables to be as given in Problem 1.32.

1.34 Find the solution of Problem 1.31 using MATLAB.

1.35 Find the solution of Problem 1.2(a) using MATLAB.

1.36 Find the solution of Problem 1.2(b) using MATLAB.