1 Infinite Sequences and Series

In experimental science and engineering, as well as in everyday life, we deal with integers, or at most rational numbers. Yet in theoretical analysis, we use real and complex numbers, as well as far more abstract mathematical constructs, fully expecting that this analysis will eventually provide useful models of natural phenomena. Hence we proceed through the construction of the real and complex numbers starting from the positive integers[1]. Understanding this construction will help the reader appreciate many basic ideas of analysis.

We start with the positive integers and zero, and introduce negative integers to allow subtraction of integers. Then we introduce *rational numbers* to permit division by integers. From arithmetic we proceed to analysis, which begins with the concept of *convergence* of infinite sequences of (rational) numbers, as defined here by the Cauchy criterion. Then we define *irrational numbers* as limits of convergent (Cauchy) sequences of rational numbers.

In order to solve algebraic equations in general, we must introduce *complex numbers* and the representation of complex numbers as points in the *complex plane*. The fundamental theorem of algebra states that every polynomial has at least one root in the complex plane, from which it follows that every polynomial of degree n has exactly n roots in the complex plane when these roots are suitably counted. We leave the proof of this theorem until we study functions of a complex variable at length in Chapter 4.

Once we understand convergence of infinite sequences, we can deal with infinite series of the form

$$\sum_{n=1}^{\infty} x_n$$

and the closely related infinite products of the form

$$\prod_{n=1}^{\infty} x_n$$

Infinite series are central to the study of solutions, both exact and approximate, to the differential equations that arise in every branch of physics. Many functions that arise in physics are defined only through infinite series, and it is important to understand the convergence properties of these series, both for theoretical analysis and for approximate evaluation of the functions.

[1]To paraphrase a remark attributed to Leopold Kronecker: "God created the positive integers; all the rest is human invention."

Introduction to Mathematical Physics. Michael T. Vaughn

ISBN: 978-3-527-40627-2

We review some of the standard tests (comparison test, ratio test, root test, integral test) for convergence of infinite series, and give some illustrative examples. We note that absolute convergence of an infinite series is necessary and sufficient to allow the terms of a series to be rearranged arbitrarily without changing the sum of the series.

Infinite sequences of functions have more subtle convergence properties. In addition to pointwise convergence of the sequence of values of the functions taken at a single point, there is a concept of *uniform convergence* on an interval of the real axis, or in a region of the complex plane. Uniform convergence guarantees that properties such as continuity and differentiability of the functions in the sequence are shared by the limit function. There is also a concept of *weak convergence*, defined in terms of the sequences of numbers generated by integrating each function of the sequence over a region with functions from a class of smooth functions (*test functions*). For example, the Dirac δ-function and its derivatives are defined in terms of weakly convergent sequences of well-behaved functions.

It is a short step from sequences of functions to consider infinite series of functions, especially *power series* of the form

$$\sum_{n=0}^{\infty} a_n z^n$$

in which the a_n are real or complex numbers and z is a complex variable. These series are central to the theory of functions of a complex variable. We show that a power series converges absolutely and uniformly inside a circle in the complex plane (the *circle of convergence*), with convergence *on* the circle of convergence an issue that must be decided separately for each particular series.

Even divergent series can be useful. We show some examples that illustrate the idea of a *semiconvergent*, or *asymptotic*, series. These can be used to determine the asymptotic behavior and approximate asymptotic values of a function, even though the series is actually divergent. We give a general description of the properties of such series, and explain Laplace's method for finding an asymptotic expansion of a function defined by an integral representation (Laplace integral) of the form

$$I(z) = \int_0^a f(t)e^{zh(t)}\,dt$$

Beyond the sequences and series generated by the mathematical functions that occur in solutions to differential equations of physics, there are sequences generated by dynamical systems themselves through the equations of motion of the system. These sequences can be viewed as *iterated maps* of the coordinate space of the system into itself; they arise in classical mechanics, for example, as successive intersections of a particle orbit with a fixed plane. They also arise naturally in population dynamics as a sequence of population counts at periodic intervals.

The asymptotic behavior of these sequences exhibits new phenomena beyond the simple convergence or divergence familiar from previous studies. In particular, there are sequences that converge, not to a single limit, but to a periodic limit cycle, or that diverge in such a way that the points in the sequence are dense in some region in a coordinate space.

An elementary prototype of such a sequence is the *logistic map* defined by

$$T_\lambda : \ x \to x_\lambda = \lambda x(1-x)$$

This map generates a sequence of points $\{x_n\}$ with

$$x_{n+1} = \lambda x_n(1-x_n)$$

$(0 < \lambda < 4)$ starting from a generic point x_0 in the interval $0 < x_0 < 1$. The behavior of this sequence as a function of the parameter λ as λ increases from 0 to 4 provides a simple illustration of the phenomena of *period doubling* and transition to *chaos* that have been an important focus of research in the past 30 years or so.

1.1 Real and Complex Numbers

1.1.1 Arithmetic

The construction of the real and complex number systems starting from the positive integers illustrates several of the structures studied extensively by mathematicians. The positive integers have the property that we can add, or we can multiply, two of them together and get a third. Each of these operations is *commutative*:

$$x \circ y = y \circ x \tag{1.1}$$

and *associative*:

$$x \circ (y \circ z) = (x \circ y) \circ z \tag{1.2}$$

(here $\circ$ denotes either addition or multiplication), but only for multiplication is there an *identity* element $\mathbf{e}$, with the property that

$$\mathbf{e} \circ x = x = x \circ \mathbf{e} \tag{1.3}$$

Of course the identity element for addition is the number zero, but zero is not a positive integer. Properties (1.2) and (1.3) are enough to characterize the positive integers as a *semigroup* under multiplication, denoted by $\mathbf{Z}_*$ or, with the inclusion of zero, a semigroup under addition, denoted by $\mathbf{Z}_+$.

Neither addition nor multiplication has an inverse defined within the positive integers. In order to define an inverse for addition, it is necessary to include zero and the negative integers. *Zero* is defined as the identity for addition, so that

$$x + 0 = x = 0 + x \tag{1.4}$$

and the *negative* integer $-x$ is defined as the *inverse* of x under addition,

$$x + (-x) = 0 = (-x) + x \tag{1.5}$$

With the inclusion of the negative integers, the equation

$$p + x = q \tag{1.6}$$

has a unique integer solution x ($\equiv q - p$) for every pair of integers p, q. Properties (1.2)–(1.5) characterize the integers as a *group* **Z** under addition, with 0 as an identity element. The fact that addition is commutative makes **Z** a *commutative*, or *Abelian*, group. The combined operations of addition with zero as identity, and multiplication satisfying Eqs. (1.2) and (1.3) with 1 as identity, characterize **Z** as a *ring*, a *commutative* ring since multiplication is also commutative. To proceed further, we need an inverse for multiplication, which leads to the introduction of *fractions* of the form p/q (with integers p, q). One important property of fractions is that they can always be reduced to a form in which the integers p, q have no common factors[2]. Numbers of this form are *rational*. With both addition and multiplication having well-defined inverses (except for division by zero, which is undefined), and the *distributive* law

$$a * (x + y) = a * x + a * c = y \tag{1.7}$$

satisfied, the rational numbers form a *field*, denoted by **Q**.

➔ **Exercise 1.1.** Let p be a prime number. Then $\sqrt{p}$ is not rational. □

Note. Here and throughout the book we use the convention that when a proposition is simply stated, the problem is to prove it, or to give a counterexample that shows it is false.

1.1.2 Algebraic Equations

The rational numbers are adequate for the usual operations of arithmetic, but to solve algebraic (polynomial) equations, or to carry out the limiting operations of calculus, we need more. For example, the quadratic equation

$$x^2 - 2 = 0 \tag{1.8}$$

has no rational solution, yet it makes sense to enlarge the rational number system to include the roots of this equation. The real *algebraic* numbers are introduced as the real roots of polynomials of any degree with integer coefficients. The algebraic numbers also form a field.

➔ **Exercise 1.2.** Show that the roots of a polynomial with rational coefficients can be expressed as roots of a polynomial with integer coefficients. □

Complex numbers are introduced in order to solve algebraic equations that would otherwise have no real roots. For example, the equation

$$x^2 + 1 = 0 \tag{1.9}$$

has no real solutions; it is "solved" by introducing the imaginary unit $i \equiv \sqrt{-1}$ so that the roots are given by $x = \pm i$. Complex numbers are then introduced as ordered pairs $(x, y) \sim$

[2] The study of properties of the positive integers, and their factorization into products of *prime* numbers, belongs to a fascinating branch of pure mathematics known as *number theory*, in which the reducibility of fractions is one of the elementary results.

$x + iy$, of real numbers; x, y can be restricted to be rational (algebraic) to define the complex rational (algebraic) numbers.

Complex numbers can be represented as points (x, y) in a plane (the *complex plane*) in a natural way, and the *magnitude* of the complex number $x + iy$ is defined by

$$|x + iy| \equiv \sqrt{x^2 + y^2} \tag{1.10}$$

In view of the identity

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.11}$$

we can also write

$$x + iy = re^{i\theta} \tag{1.12}$$

with $r = |x + iy|$ and $\tan\theta = y/x$. These relations have an obvious interpretation in terms of the polar coordinates of the point (x, y). We also define

$$\arg\, z \equiv \theta \tag{1.13}$$

for $z \neq 0$. The angle $\arg\, z$ is the *phase* of z. Evidently it can only be defined as $\mathrm{mod}\ 2\pi$; adding any integer multiple of 2π to $\arg\, z$ does not change the complex number z, since

$$e^{2\pi i} = 1 \tag{1.14}$$

Equation (1.14) is one of the most remarkable equations of mathematics.

1.1.3 Infinite Sequences; Irrational Numbers

To complete the construction of the real and complex numbers, we need to look at some elementary properties of sequences, starting with the formal definitions:

Definition 1.1. A *sequence* of numbers (real or complex) is an ordered set of numbers in one-to-one correspondence with the positive integers; write $\{z_n\} \equiv \{z_1, z_2, \ldots\}$. ∎

Definition 1.2. The sequence $\{z_n\}$ is *bounded* if there is some positive number M such that $|z_n| < M$ for all positive integers n. ∎

Definition 1.3. The sequence $\{x_n\}$ of real numbers is *increasing* (*decreasing*) if $x_{n+1} > x_n$ ($x_{n+1} < x_n$) for every n. The sequence is *nondecreasing* (*nonincreasing*) if $x_{n+1} \geq x_n$ ($x_{n+1} \leq x_n$) for every n. A sequence belonging to one of these classes is *monotone* (or *monotonic*). ∎

Remark. The preceding definition is restricted to real numbers because it is only for real numbers that we can define a "natural" ordering that is compatible with the standard measure of the distance between the numbers. □

Definition 1.4. The sequence $\{z_n\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there is a positive integer N such that $|z_p - z_q| < \varepsilon$ whenever $p, q > N$. ∎

Definition 1.5. The sequence $\{z_n\}$ is *convergent* to the *limit* z (write $\{z_n\} \to z$) if for every $\varepsilon > 0$ there is a positive integer N such that $|z_n - z| < \varepsilon$ whenever $n > N$. ∎

There is no guarantee that a Cauchy sequence of rational numbers converges to a rational, or even algebraic, limit. For example, the sequence $\{x_n\}$ defined by

$$x_n \equiv \left(1 + \frac{1}{n}\right)^n \tag{1.15}$$

converges to the limit $e = 2.71828\ldots$, the base of natural logarithms. It is true, though nontrivial to prove, that e is not an algebraic number. A real number that is not algebraic is *transcendental*. Another famous transcendental number is π, which is related to e through Eq. (1.14).

If we want to insure that every Cauchy sequence of rational numbers converges to a limit, we must include the *irrational numbers*, which can be *defined* as limits of Cauchy sequences of rational numbers. As examples of such sequences, imagine the infinite, nonterminating, nonperiodic decimal expansions of transcendental numbers such as e or π, or algebraic numbers such as $\sqrt{2}$. Countless computer cycles have been used in calculating the digits in these expansions.

The set of *real* numbers, denoted by $\mathbf{R}$, can now be defined as the set containing rational numbers together with the limits of Cauchy sequences of rational numbers. The set of *complex* numbers, denoted by $\mathbf{C}$, is then introduced as the set of all ordered pairs $(x, y) \sim x+iy$ of real numbers. Once we know that every Cauchy sequence of real (or rational) numbers converges to a real number, it is a simple exercise to show that every Cauchy sequence of complex numbers converges to a complex number.

Monotonic sequences are especially important, since they appear as partial sums of infinite series of positive terms. The key property is contained in the

Theorem 1.1. A monotonic sequence $\{x_n\}$ is convergent if and only if it is bounded.

Proof. If the sequence is unbounded, it will diverge to $\pm\infty$, which simply means that for any positive number M, no matter how large, there is an integer N such that $x_n > M$ (or $x_n < -M$ if the sequence is monotonic nonincreasing) for any $n \geq N$. This is true, since for any positive number M, there is at least one member x_N of the sequence with $x_N > M$ (or $x_N < -M$)—otherwise M would be a bound for the sequence—and hence $x_n > M$ (or $x_n < -M$) for any $n \geq N$ in view of the monotonic nature of the sequence. ■

If the monotonic nondecreasing sequence $\{x_n\}$ is bounded from above, then in order to have a limit, there must be a bound that is smaller than any other bound (such a bound is the *least upper bound* of the sequence). If the sequence has a limit X, then X is certainly the least upper bound of the sequence, while if a least upper bound $\overline{X}$ exists, then it must be the limit of the sequence. For if there is some $\varepsilon > 0$ such that $\overline{X} - x_n > \varepsilon$ for all n, then $\overline{X} - \varepsilon$ will be an upper bound to the sequence smaller than $\overline{X}$.

The existence of a least upper bound is intuitively plausible, but its existence cannot be proven from the concepts we have introduced so far. There are alternative axiomatic formulations of the real number system that guarantee the existence of the least upper bound; the convergence of any bounded monotonic nondecreasing sequence is then a consequence as just explained. The same argument applies to bounded monotonic nonincreasing sequences, which must then have a *greatest lower bound* to which the sequence converges.

1.1.4 Sets of Real and Complex Numbers

We also need some elementary definitions and results about sets of real and complex numbers that are generalized later to other structures.

Definition 1.6. For real numbers, we can define an *open* interval:

$$(a, b) \equiv \{x|\, a < x < b\}$$

or a *closed* interval:

$$[a, b] \equiv \{x|\, a \leq x \leq b\}$$

as well as *semiopen* (or *semiclosed*) intervals:

$$(a, b] \equiv \{x|\, a < x \leq b\} \quad \text{and} \quad [a, b) \equiv \{x|\, a \leq x < b\}$$

A *neighborhood* of the real number x_0 is any open interval containing x_0. An ε-*neighborhood* of x_0 is the set of all points x such that

$$|x - x_0| < \varepsilon \tag{1.16}$$

This concept has an obvious extension to complex numbers: An (ε)-*neighborhood* of the complex number z_0, denoted by $N_\varepsilon(z_0)$, is the set of all points z such that

$$0 < |z - z_0| < \varepsilon \tag{1.17}$$

Note that for complex numbers, we exclude the point z_0 from the neighborhood $N_\varepsilon(z_0)$. ∎

Definition 1.7. The set $\mathcal{S}$ of real or complex numbers is *open* if for every x in $\mathcal{S}$, there is a neighborhood of x lying entirely in $\mathcal{S}$. $\mathcal{S}$ is *closed* if its complement is open. $\mathcal{S}$ is *bounded* if there is some positive M such that $x < M$ for every x in $\mathcal{S}$ (M is then a *bound* of $\mathcal{S}$). ∎

Definition 1.8. x is a *limit point* of the set $\mathcal{S}$ if every neighborhood of x contains at least one point of $\mathcal{S}$. ∎

While x itself need not be a member of the set $\mathcal{S}$, this definition implies that every neighborhood of x in fact contains an infinite number of points of $\mathcal{S}$. An alternative definition of a closed set can be given in terms of limit points, and one of the important results of analysis is that every bounded infinite set contains at least one limit point.

➔ **Exercise 1.3.** Show that the set $\mathcal{S}$ of real or complex numbers is closed if and only if every limit point of $\mathcal{S}$ is an element of $\mathcal{S}$. □

➔ **Exercise 1.4.** (Bolzano–Weierstrass theorem) Every bounded infinite set of real or complex numbers contains at least one limit point. □

Definition 1.9. The set $\mathcal{S}$ is *everywhere dense*, or simply *dense*, in a region $\mathcal{R}$ if there is at least one point of $\mathcal{S}$ in any neighborhood of every point in $\mathcal{R}$. ∎

❑ **Example 1.1.** The set of rational numbers is everywhere dense on the real axis. ∎

1.2 Convergence of Infinite Series and Products

1.2.1 Convergence and Divergence; Absolute Convergence

If $\{z_k\}$ is a sequence of numbers (real or complex), the formal sum

$$S \equiv \sum_{k=1}^{\infty} z_k \tag{1.18}$$

is an *infinite series*, whose *partial sums* are defined by

$$s_n \equiv \sum_{k=1}^{n} z_k \tag{1.19}$$

The series $\sum z_k$ is *convergent* (to the *value* s) if the sequence $\{s_n\}$ of partial sums converges to s, otherwise *divergent*. The series is *absolutely convergent* if the series $\sum |z_k|$ is convergent; a series that is convergent but not absolutely convergent is *conditionally convergent*. Absolute convergence is an important property of a series, since it allows us to rearrange terms of the series without altering its value, while the sum of a conditionally convergent series can be changed by reordering it (this is proved later on).

➔ **Exercise 1.5.** If the series $\sum z_k$ is convergent, then the sequence $\{z_k\} \to 0$. □

➔ **Exercise 1.6.** If the series $\sum z_k$ is absolutely convergent, then it is convergent. □

To study absolute convergence, we need only consider a series $\sum x_k$ of positive real numbers ($\sum |z_k|$ is such a series). The sequence of partial sums of a series of positive real numbers is obviously nondecreasing. From the theorem on monotonic sequences in the previous section then follows

Theorem 1.2. The series $\sum x_k$ of positive real numbers is convergent if and only if the sequence of its partial sums is bounded.

❑ **Example 1.2.** Consider the *geometric series*

$$S(x) \equiv \sum_{k=0}^{\infty} x^k \tag{1.20}$$

for which the partial sums are given by

$$s_n = \sum_{k=0}^{n} x^k = \frac{1-x}{1-x^{n+1}} \tag{1.21}$$

These partial sums are bounded if $0 \leq x < 1$, in which case

$$\{s_n\} \;\to\; \frac{1}{1-x} \tag{1.22}$$

The series diverges for $x \geq 1$. The corresponding series

$$S(z) \equiv \sum_{k=0}^{\infty} z^k \tag{1.23}$$

for complex z is then absolutely convergent for $|z| < 1$, divergent for $|z| > 1$. The behavior on the *unit circle* $|z| = 1$ in the complex plane must be determined separately (the series actually diverges everywhere on the circle since the sequence $\{z^k\} \not\to 0$; see Exercise 1.5). ∎

Remark. We will see that the function $S(z)$ defined by the series (1.23) for $|z| < 1$ can be defined to be $1/(1-z)$ for complex $z \neq 1$, even outside the region of convergence of the series, using the properties of $S(z)$ as a function of the complex variable z. This is an example of a procedure known as *analytic continuation*, to be explained in Chapter 4. □

❑ **Example 1.3.** The *Riemann* ζ*-function* is defined by

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.24}$$

The series for $\zeta(s)$ with $s = \sigma + i\tau$ is absolutely convergent if and only if the series for $\zeta(\sigma)$ is convergent. Denote the partial sums of the latter series by

$$s_N(\sigma) = \sum_{n=1}^{N} \frac{1}{n^\sigma} \tag{1.25}$$

Then for $\sigma \leq 1$ and $N \geq 2^m$ (m integer), we have

$$s_N(\sigma) \geq s_N(1) \geq s_{2^m}(1) > s_{2^{m-1}}(1) + \frac{1}{2} > \cdots > \frac{m}{2} \tag{1.26}$$

Hence the sequence $\{s_N(\sigma)\}$ is unbounded and the series diverges. Note that for $s = 1$, Eq. (1.24) is the *harmonic series*, which is shown to diverge in elementary calculus courses. On the other hand, for $\sigma > 1$ and $N \leq 2^m$ with m integer, we have

$$\begin{aligned} s_N(\sigma) &< s_{2^m}(\sigma) < s_{2^{m-1}}(\sigma) + \left(\frac{1}{2}\right)^{(m-1)(\sigma-1)} < \cdots \\ &< \sum_{k=0}^{m-1} \left(\frac{1}{2}\right)^{k(\sigma-1)} < \frac{1}{1-2^{(1-\sigma)}} \end{aligned} \tag{1.27}$$

Thus the sequence $\{s_N(\sigma)\}$ is bounded and hence converges, so that the series (1.24) for $\zeta(s)$ is absolutely convergent for $\sigma = \text{Re } s > 1$. Again, we will see in Chapter 4 that $\zeta(s)$ can be defined for complex s beyond the range of convergence of the series (1.24) by analytic continuation. ∎

1.2.2 Tests for Convergence of an Infinite Series of Positive Terms

There are several standard tests for convergence of a series of positive terms:

Comparison test. Let $\sum x_k$ and $\sum y_k$ be two series of positive numbers, and suppose that for some integer $N > 0$ we have $y_k \leq x_k$ for all $k > N$. Then

(i) if $\sum x_k$ is convergent, $\sum y_k$ is also convergent, and

(ii) if $\sum y_k$ is divergent, $\sum x_k$ is also divergent.

This is fairly obvious, but to give a formal proof, let $\{s_n\}$ and $\{t_n\}$ denote the sequences of partial sums of $\sum x_k$ and $\sum y_k$, respectively. If $y_k \leq x_k$ for all $k > N$, then

$$t_n - t_N \leq s_n - s_N$$

for all $n > N$. Thus if $\{s_n\}$ is bounded, then $\{t_n\}$ is bounded, and if $\{t_n\}$ is unbounded, then $\{s_n\}$ is unbounded.

Remark. The comparison test has been used implicitly in the discussion of the ζ-function to show the absolute convergence of the series 1.24 for $\sigma = \text{Re}\, s > 1$. □

Ratio test. Let $\sum x_k$ be a series of positive numbers, and let $r_k \equiv x_{k+1}/x_k$ be the ratios of successive terms. Then

(i) if only a finite number of $r_k > a$ for some a with $0 < a < 1$, then the series converges, and

(ii) if only a finite number of $r_k < 1$, then the series diverges.

In case (i), only a finite number of the r_k are larger than a, so there is some positive M such that $x_k < Ma^k$ for all k, and the series converges by comparison with the geometric series. In case (ii), the series diverges since the individual terms of the series do not tend to zero.

Remark. The ratio test works if the largest limit point of the sequence $\{r_k\}$ is either greater than 1 or smaller than 1. If the largest limit point is exactly equal to 1, then the ratio test does not answer the question of convergence, as seen by the example of the ζ-function series (1.24). □

Root test. Let $\sum x_k$ be a series of positive numbers, and let $\varrho_k \equiv \sqrt[k]{x_k}$. Then

(i) if only a finite number of $\varrho_k > a$ for some positive $a < 1$, then the series converges, and

(ii) if infinitely many $\varrho_k > 1$, the series diverges.

As with the ratio test, we can construct a comparison with the geometric series. In case (i), only a finite number of roots ϱ_k are bigger than a, so there is some positive M such that $x_k < Ma^k$ for all k, and the series converges by comparison with the geometric series. In case (ii), the series diverges since the individual terms of the series do not tend to zero.

Remark. The root test, like the ratio test, works if the largest limit point of the sequence $\{\varrho_k\}$ is either greater than 1 or smaller than 1, but fails to decide convergence if the largest limit point is exactly equal to 1. □

Integral test. Let $f(t)$ be a continuous, positive, and nonincreasing function for $t \geq 1$, and let $x_k \equiv f(k)$ $(k = 1, 2, \ldots)$. Then $\sum x_k$ converges if and only if the integral

$$I \equiv \int_1^\infty f(t)\, dt < \infty \tag{1.28}$$

also converges. To show this, note that

$$\int_k^{k+1} f(t)\,dt \le x_k \le \int_{k-1}^{k} f(t)\,dt \tag{1.29}$$

which is easy to see by drawing a graph. The partial sums s_n of the series then satisfy

$$\int_1^{n+1} f(t)\,dt \le s_n = \sum_{k=1}^{n} x_k \le x_1 + \int_1^{n} f(t)\,dt \tag{1.30}$$

and are bounded if and only if the integral (1.28) converges.

Remark. If the integral (1.28) converges, it provides a (very) rough estimate of the value of the infinite series, since

$$\int_{N+1}^{\infty} f(t)\,dt \;\le\; s - s_N \;=\; \sum_{k=N+1}^{\infty} x_k \;\le\; \int_N^{\infty} f(t)\,dt \tag{1.31}$$

1.2.3 Alternating Series and Rearrangements

In addition to a series of positive terms, we consider an *alternating series* of the form

$$S \equiv \sum_{k=0}^{\infty} (-1)^k x_k \tag{1.32}$$

with $x_k > 0$ for all k. Here there is a simple criterion (due to Leibnitz) for convergence: if the sequence $\{x_k\}$ is nonincreasing, then the series S converges if and only if $\{x_k\} \to 0$, and if S converges, its value lies between any two successive partial sums. This follows from the observation that for any n the partial sums s_n of the series (1.32) satisfy

$$s_{2n+1} < s_{2n+3} < \cdots < s_{2n+2} < s_{2n} \tag{1.33}$$

❑ **Example 1.4.** The alternating harmonic series

$$A \equiv 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \tag{1.34}$$

is convergent according to this criterion, even though it is not absolutely convergent (the series of absolute values is the harmonic series we have just seen to be divergent). In fact, evaluating the logarithmic series (Eq. (1.69) below) for $z = 1$ shows that $A = \ln 2$. ∎

Is there any significance of the ordering of terms in an infinite series? The short answer is that terms can be rearranged at will in an absolutely convergent series without changing the value of the sum, while changing the order of terms in a conditionally convergent series can change its value, or even make it diverge.

Definition 1.10. If $\{n_1, n_2, \ldots\}$ is a permutation of $\{1, 2, \ldots\}$, then the sequence $\{\zeta_k\}$ is a *rearrangement* of $\{z_k\}$ if

$$\zeta_k = z_{n_k} \tag{1.35}$$

for every k. Then also the series $\sum \zeta_k$ is a rearrangement of $\sum z_k$. ▮

❑ **Example 1.5.** The alternating harmonic series (1.34) can be rearranged in the form

$$A' = \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \cdots \tag{1.36}$$

which is still a convergent series, but its value is not the same as that of A (see below). ▮

Theorem 1.3. If the series $\sum z_k$ is absolutely convergent, and $\sum \zeta_k$ is a rearrangement of $\sum z_k$, then $\sum \zeta_k$ is absolutely convergent.

Proof. Let $\{s_n\}$ and $\{\sigma_n\}$ denote the sequences of partial sums of $\sum z_k$ and $\sum \zeta_k$, respectively. If $\varepsilon > 0$, choose N such that $|s_n - s_m| < \varepsilon$ for all $n, m > N$, and let $Q \equiv \max\{n_1, \ldots, n_N\}$. Then $|\sigma_n - \sigma_m| < \varepsilon$ for all $n, m > Q$. ■

On the other hand, if a series in not absolutely convergent, then its value can be changed (almost at will) by rearrangement of its terms. For example, the alternating series in its original form (1.34) can be expressed as

$$A = \sum_{n=0}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} \tag{1.37}$$

This is an absolutely convergent series of positive terms whose value is $\ln 2 = 0.693\ldots$, as already noted. On the other hand, the rearranged series (1.36) can be expressed as

$$A' = \sum_{n=0}^{\infty} \left(\frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}\right) = \sum_{n=0}^{\infty} \frac{8n+5}{2(n+1)(4n+1)(4n+3)} \tag{1.38}$$

which is another absolutely convergent series of positive terms. Including just the first term of this series shows that

$$A' > \frac{5}{6} > \ln 2 = A \tag{1.39}$$

In fact, any series that is not absolutely convergent can be rearranged into a divergent series.

Theorem 1.4. If the series $\sum x_k$ of real terms is conditionally convergent, then there is a divergent rearrangement of $\sum x_k$.

Proof. Let $\{\xi_1, \xi_2, \ldots\}$ be the sequence of positive terms in $\{x_k\}$, and $\{-\eta_1, -\eta_2, \ldots\}$ be the sequence of negative terms. Then at least one of the series $\sum \xi_k$, $\sum \eta_k$ is divergent (otherwise the series would be absolutely convergent). Suppose $\sum \xi_k$ is divergent. Then we can choose a sequence $n_1, n_2, \ldots$ such that

$$\sum_{k=n_m}^{n_{m+1}-1} \xi_k > 1 + \eta_m \tag{1.40}$$

($m = 1, 2, \ldots$), and the rearranged series

$$S' \equiv \sum_{k=n_1}^{n_2-1} \xi_k - \eta_1 + \sum_{k=n_2}^{n_3-1} \xi_k - \eta_2 + \cdots \tag{1.41}$$

is divergent. ■

Remark. It follows as well that a conditionally convergent series $\sum z_k$ of complex terms must have a divergent rearrangement. For if $z_k = x_k + iy_k$, then either $\sum x_k$ or $\sum y_k$ is conditionally convergent, and hence has a divergent rearrangement. □

1.2.4 Infinite Products

Closely related to infinite series are *infinite products* of the form

$$\prod_{m=1}^{\infty} (1 + z_m) \tag{1.42}$$

($\{z_m\}$ is a sequence of complex numbers), with *partial products*

$$p_n \equiv \prod_{m=1}^{n} (1 + z_k) \tag{1.43}$$

The product $\prod (1+z_m)$ is *convergent* (to the *value* p) if the sequence $\{p_n\}$ of partial products converges to $p \neq 0$, *convergent to zero* if a finite number of factors are 0, *divergent to zero* if $\{p_n\} \to 0$ with no vanishing p_n, and *divergent* if $\{p_n\}$ is divergent. The product is *absolutely convergent* if $\prod (1 + |z_m|)$ is convergent; a product that is convergent but not absolutely convergent is *conditionally convergent*.

The absolute convergence of a product is simply related to the absolute convergence of a related series: if $\{x_m\}$ is a sequence of positive real numbers, then the product $\prod (1 + x_m)$ is convergent if and only if the series $\sum x_m$ is convergent. This follows directly from the observation

$$\sum_{m=1}^{n} x_m < \prod_{m=1}^{n} (1 + x_m) < \exp\left(\sum_{m=1}^{n} x_m\right) \tag{1.44}$$

Also, the product $\prod (1-x_m)$ is convergent if and only if the series $\sum x_m$ is convergent (show this).

❑ **Example 1.6.** Consider the infinite product

$$P \equiv \prod_{m=2}^{\infty} \left(\frac{m^3 - 1}{m^3 + 1}\right) < \prod_{m=2}^{\infty} \left(1 - \frac{1}{m^3}\right) \tag{1.45}$$

The product is (absolutely) convergent, since the series

$$\sum_{m=1}^{\infty} \frac{1}{m^3} = \zeta(3)$$

is convergent. Evaluation of the product is left as a problem. ■

1.3 Sequences and Series of Functions

1.3.1 Pointwise Convergence and Uniform Convergence of Sequences of Functions

Questions of convergence of sequences and series of functions in some domain of variables can be answered at each point by the methods of the preceding section. However, the issues of continuity and differentiability of the limit function require more care, since the limiting procedures involved approaching a point in the domain need not be interchangeable with passing to the limit of the sequence or series (convergence of an infinite series of functions is defined in the usual way in terms of the convergence of the sequence of partial sums of the series). Thus we introduce

Definition 1.11. The sequence $\{f_n(z)\}$ of functions of the variable z (real or complex) is (*pointwise*) *convergent* to the function $f(z)$ in the region $\mathcal{R}$:

$$\{f_n(z)\} \to f(z) \text{ in } \mathcal{S}$$

if the sequence $\{f_n(z_0)\} \to f(z_0)$ at every point z_0 in $\mathcal{R}$.

Definition 1.12. $\{f_n(z)\}$ is *uniformly convergent* to $f(z)$ in the closed, bounded $\mathcal{R}$:

$$\{f_n(z)\} \rightrightarrows f(z) \text{ in } \mathcal{S}$$

if for every $\varepsilon > 0$ there is a positive integer N such that $|f_n(z) - f(z)| < \varepsilon$ for every $n > N$ and *every* point z in $\mathcal{R}$. ∎

Remark. Note the use of different arrow symbols ($\to$ and $\Rightarrow$) to denote strong and uniform convergence, as well as the symbol ($\rightharpoonup$) introduced below to denote weak convergence. □

❑ **Example 1.7.** Consider the sequence $\{x^n\}$. Evidently $\{x^n\} \to 0$ for $0 \le x < 1$. Also, the sequence $\{x^n\} \Rightarrow 0$ on any closed interval $0 \le x \le 1 - \delta$ $(0 < \delta < 1)$, since for any such x, we have $|x^n| < \varepsilon$ for all $n > N$ if N is chosen so that $|1-\delta|^N < \varepsilon$. However, we cannot say that the sequence is uniformly convergent on the open interval $0 < x < 1$, since if $0 < \varepsilon < 1$ and n is any positive integer, we can find some x in (0, 1) such that $x^n > \varepsilon$. The point here is that to discuss uniform convergence, we need to consider a region that is closed and bounded, with no limit point at which the series is divergent. ∎

It is one of the standard theorems of advanced calculus that properties of continuity of the elements of a uniformly convergent sequence are shared by the limit of the sequence. Thus if $\{f_n(z)\} \Rightarrow f(z)$ in the region $\mathcal{R}$, and if each of the $f_n(z)$ is continuous in the closed bounded region $\mathcal{R}$, then the limit function $f(z)$ is also continuous in $\mathcal{R}$. Differentiability requires a separate check that the sequence of derivative functions $\{f_n'(z)\}$ is convergent, since it may not be. If the sequence of derivatives actually is uniformly convergent, then it converges to the derivative of the limit function $f(z)$.

❑ **Example 1.8.** Consider the function $f(z)$ defined by the series

$$f(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} \sin n^2 \pi z \tag{1.46}$$

This series is absolutely and uniformly convergent on the entire real axis, since it is bounded by the convergent series

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{1.47}$$

However, the formal series

$$f'(z) \equiv \pi \sum_{n=1}^{\infty} \cos n^2 \pi z \tag{1.48}$$

converges nowhere, since the terms in the series do not tend to zero for large n. A similar example is the series

$$g(z) \equiv \sum_{n=1}^{\infty} a^n \sin 2^n \pi z \tag{1.49}$$

for which the convergence properties of the derivative can be worked out as an exercise. Functions of this type were introduced as illustrative examples by Weierstrass. ∎

1.3.2 Weak Convergence; Generalized Functions

There is another type of convergent sequence, whose limit is not a function in the classical sense, but which defines a kind of generalized function widely used in physics. Suppose $\mathcal{C}$ is a class of well-behaved functions (*test functions*) on a region $\mathcal{R}$–typically functions that are continuous with continuous derivatives of suitably high order. Then the sequence of functions $\{f_n(z)\}$ (that need not themselves be in $\mathcal{C}$) is *weakly convergent* (relative to $\mathcal{C}$) if the sequence

$$\left\{ \int_{\mathcal{R}} f_n(z)\, g(z)\, d\tau \right\} \tag{1.50}$$

is convergent for every function $g(z)$ in the class $\mathcal{C}$. The limit of a weakly convergent sequence is a *generalized function*, or *distribution*. It need not have a value at every point of $\mathcal{R}$. If

$$\left\{ \int_{\mathcal{R}} f_n(z)\, g(z)\, d\tau \right\} \to \int_{\mathcal{R}} f(z)\, g(z)\, d\tau \tag{1.51}$$

for every $g(z)$ in $\mathcal{C}$, then $\{f_n(z)\} \rightharpoonup f(z)$ (the symbol $\rightharpoonup$ denotes weak convergence), but the weak convergence need not define the value of the limit $f(z)$ at discrete points.

❑ **Example 1.9.** Consider the sequence $\{f_n(x)\}$ defined by

$$f_n(x) = \begin{cases} \dfrac{n}{2} & -\dfrac{1}{n} \le x \le \dfrac{1}{n} \\ 0\,, & \text{otherwise} \end{cases} \tag{1.52}$$

Then $\{f_n(x)\} \to 0$ for every $x \neq 0$, but

$$\int_{-\infty}^{\infty} f_n(x)\,dx = 1 \tag{1.53}$$

for $n = 1, 2, \ldots$, and, if $g(x)$ is continuous at $x = 0$,

$$\left\{\int_{-\infty}^{\infty} f_n(x)\,g(x)\,dx\right\} \to g(0) \tag{1.54}$$

The weak limit of the sequence $\{f_n(x)\}$ thus has the properties attributed to the *Dirac* δ-*function* $\delta(x)$, defined here as a distribution on the class of functions continuous at $x = 0$. The derivative of the δ-function can be defined as a generalized function on the class of functions with continuous derivative at $x = 0$ using integration by parts to write

$$\int_{-\infty}^{\infty} \delta'(x)\,g(x)\,dx = -\int_{-\infty}^{\infty} \delta(x)\,g'(x)\,dx = -g'(0) \tag{1.55}$$

Similarly, the nth derivative of the δ-function is defined as a generalized function on the class of functions with the continuous nth derivative at $x = 0$ by

$$\begin{aligned} \int_{-\infty}^{\infty} \delta^{(n)}(x)\,g(x)\,dx &= -\int_{-\infty}^{\infty} \delta^{(n-1)}(x)\,g'(x)\,dx \\ &= \cdots = (-1)^n\,g^{(n)}(0) \end{aligned} \tag{1.56}$$

using repeated integration by parts. ∎

1.3.3 Infinite Series of Functions; Power Series

Convergence properties of infinite series

$$\sum_{k=0}^{\infty} f_k(z)$$

of functions are identified with those of the corresponding sequence

$$s_n(z) \equiv \sum_{k=0}^{n} f_k(z) \tag{1.57}$$

of partial sums. The series $\sum_k f_k(z)$ is (pointwise, uniformly, weakly) convergent on $\mathcal{R}$ if the sequence $\{s_n(z)\}$ is (pointwise, uniformly, weakly) convergent on $\mathcal{R}$, and absolutely convergent if the sum of absolute values,

$$\sum_k |f_k(z)|$$

is convergent.

An important class of infinite series of functions is the *power series*

$$S(z) \equiv \sum_{n=0}^{\infty} a_n z^n \tag{1.58}$$

in which $\{a_n\}$ is a sequence of complex numbers and z a complex variable. The basic convergence properties of power series are contained in

Theorem 1.5. Let $S(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ be a power series, $\alpha_n \equiv \sqrt[n]{|a_n|}$, and let α be the largest limit point of the sequence $\{\alpha_n\}$. Then

(i) If $\alpha = 0$, then the series $S(z)$ is absolutely convergent for all z, and uniformly on any bounded region of the complex plane,

(ii) If α does not exist ($\alpha = \infty$), then $S(z)$ is divergent for any $z \neq 0$,

(iii) If $0 < \alpha < \infty$, then $S(z)$ is absolutely convergent for $|z| < r \equiv 1/\alpha$, uniformly within any circle $|z| \leq \rho < r$, and $S(z)$ is divergent for $|z| > r$.

Proof. Since $\sqrt[n]{|a_n z^n|} = \alpha_n |z|$, results (i)–(iii) follow directly from the root test. ■

Thus the region of convergence of a power series is at least the interior of a circle in the complex plane, the *circle of convergence*, and r is the *radius of convergence*. Note that convergence tests other than the root test can be used to determine the radius of convergence of a given power series. The behavior of the series *on* the circle of convergence must be determined separately for each series; various possibilities are illustrated in the examples and problems.

Now suppose we have a function $f(z)$ defined by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.59}$$

with the radius of convergence $r > 0$. Then $f(z)$ is differentiable for $|z| < r$, and its derivative is given by the series

$$f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} \, z^n \tag{1.60}$$

which is absolutely convergent for $|z| < r$ (show this). Thus a power series can be differentiated term by term inside its circle of convergence. Furthermore, $f(z)$ is differentiable to any order for $|z| < r$, and the kth derivative is given by the series

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} \, a_{n+k} \, z^n \tag{1.61}$$

since this series is also absolutely convergent for $|z| < r$. It follows that

$$a_k = f^{(k)}(0)/k! \tag{1.62}$$

Thus every power series with positive radius of convergence is a *Taylor series* defining a function with derivatives of any order. Such functions are *analytic functions*, which we study more deeply in Chapter 4.

❑ **Example 1.10.** Following are some standard power series; it is a useful exercise to verify the radius of convergence for each of these power series using the tests given here. ∎

(i) The *binomial series* is

$$(1+z)^\alpha \equiv \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \tag{1.63}$$

where

$$\binom{\alpha}{n} \equiv \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \frac{\Gamma(\alpha+1)}{n!\,\Gamma(\alpha-n+1)} \tag{1.64}$$

is the *generalized binomial coefficient*. Here $\Gamma(z)$ is the Γ-function that generalizes the elementary factorial function; it is discussed at length in Appendix A. For $\alpha = m = 0, 1, 2, \ldots$, the series terminates after $m+1$ terms and thus converges for all z; otherwise, note that

$$\binom{\alpha}{n+1} \Big/ \binom{\alpha}{n} = \frac{\alpha-n}{n+1} \longrightarrow -1 \tag{1.65}$$

whence the series (1.63) has the radius of convergence $r = 1$.

(ii) The *exponential series*

$$e^z \equiv \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1.66}$$

has infinite radius of convergence.

(iii) The *trigonometric functions* are given by the power series

$$\sin z \equiv \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{1.67}$$

$$\cos z \equiv \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \tag{1.68}$$

with infinite radius of convergence.

(iv) The *logarithmic series*

$$\ln(1+z) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} \tag{1.69}$$

has the radius of convergence $r = 1$.

(v) The *arctangent series*

$$\tan^{-1} z \equiv \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} \tag{1.70}$$

has the radius of convergence $r = 1$.

1.4 Asymptotic Series

1.4.1 The Exponential Integral

Consider the function $E_1(z)$ defined by

$$E_1(z) \equiv \int_z^\infty \frac{e^{-t}}{t}\,dt = e^{-z}\int_0^\infty \frac{e^{-u}}{u+z}\,du \equiv e^{-z}\,I(z) \tag{1.71}$$

$E_1(z)$ is the *exponential integral*, a tabulated function. An expansion of $E_1(z)$ about $z = 0$ is given by

$$\begin{aligned} E_1(z) &= \int_1^\infty \frac{e^{-t}}{t}\,dt - \int_1^z \frac{e^{-t}}{t}\,dt = -\ln z + \int_1^z \frac{1-e^{-t}}{t}\,dt + \int_1^\infty \frac{e^{-t}}{t}\,dt \\ &= -\ln z - \left[\int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_1^\infty \frac{e^{-t}}{t}\,dt\right] - \sum_{n=1}^\infty \frac{(-1)^n}{n}\,\frac{z^n}{n!} \end{aligned} \tag{1.72}$$

Here the term in the square brackets is the *Euler–Mascheroni* constant $\gamma = 0.5772\ldots$, and the power series has infinite radius of convergence.

Suppose now $|z|$ is large. Then the series (1.72) converges slowly, and a better estimate of the integral $I(z)$ can be obtained by introducing the expansion

$$\frac{1}{u+z} = \frac{1}{z}\sum_{n=0}^\infty (-1)^n \left(\frac{u}{z}\right)^n \tag{1.73}$$

into the integral (1.71). Then term-by-term integration leads to the series expansion

$$I(z) = \sum_{n=0}^\infty (-1)^n \frac{n!}{z^{n+1}} \tag{1.74}$$

Unfortunately, the formal power series (1.74) diverges for all finite z. This is due to the fact that the series expansion (1.73) of $1/(u+z)$ is not convergent over the entire range of integration $0 \le u < \infty$. However, the main contribution to the integral comes from the region of small u, where the expansion does converge, and, in fact, the successive terms of the series for $I(z)$ decrease in magnitude for $n+1 \le |z|$; only for $n+1 > |z|$ do they begin to diverge. This suggests that the series (1.74) might provide a useful approximation to the integral $I(z)$ if appropriately truncated.

To obtain an estimate of the error in truncating the series, note that repeated integration by parts in Eq. (1.71) gives

$$\begin{aligned} I(z) &= \sum_{n=0}^N (-1)^n \frac{n!}{z^{n+1}} + (-1)^{N+1}(N+1)!\int_0^\infty \frac{e^{-u}}{(u+z)^{N+2}}\,du \\ &\equiv S_N(z) + R_N(z) \end{aligned} \tag{1.75}$$

If $\mathrm{Re}\, z > 0$, then we can bound the remainder term $R_N(z)$ by

$$|R_N(z)| \le \frac{(N+1)!}{|z|^{N+2}} \tag{1.76}$$

since $|u+z| \ge |z|$ for all $u \ge 0$ when $\mathrm{Re}\, z \ge 0$. Hence the remainder term $R_N(z) \to 0$ as $z \to \infty$ in the right half of the complex plane, so that $I(z)$ can be approximated by $S_N(z)$ with a relative error that vanishes as $z \to \infty$ in the right half-plane. In fact, when $\mathrm{Re}\, z < 0$ we have

$$|R_N\,(z)| \le \frac{(N+1)!}{|\mathrm{Im}\, z|^{N+2}} \tag{1.77}$$

so that the series (1.74) is valid in any sector $-\delta \le \arg\ z \le \delta$ with $0 < \delta < \pi$.Note also that for fixed z, $|R_N(z)|$ has a minimum for $N+1 \cong |z|$, so that we can obtain a "best" estimate of $I(z)$ by truncating the expansion after about $N+1$ terms.

The series (1.74) is an *asymptotic* (or *semiconvergent*) series for the function $I(z)$ defined by Eq. (1.71). Asymptotic series are useful, often more useful than convergent series, in exhibiting the behavior of functions such as solutions to differential equations, for limiting values of their arguments. An asymptotic series can also provide a practical method for evaluating a function, even though it can never give the "exact" value of the function because it is divergent. The device of integration by parts, for which the illegal power series expansion of the integrand is a shortcut, is one method of generating an asymptotic series. Watson's lemma, introduced below, provides another.

1.4.2 Asymptotic Expansions; Asymptotic Series

Before looking at more examples, we introduce some standard terminology associated with asymptotic expansions.

Definition 1.13. $f(z)$ is *of order* $g(z)$ as $z \to z_0$, or $f(z) = O[g(z)]$ as $z \to z_0$, if there is some positive M such that $|f(z)| \le M|g(z)|$ in some neighborhood of z_0. Also, $f(z) = o[g(z)]$ (read "$f(z)$ is *little o* $g(z)$") as $z \to z_0$ if

$$\lim_{z \to z_0} f(z)/g(z) = 0 \tag{1.78}$$

❑ **Example 1.11.** We have

(i) $z^{n+1} = o(z^n)$ as $z \to 0$ for any n.

(ii) $e^{-z} = o(z^n)$ for any n as $z \to \infty$ in the right half of the complex plane.

(iii) $E_1(z) = O(e^{-z}/z)$, or $E_1(z) = o(e^{-z})$, as $z \to \infty$ in any sector $-\delta \le \arg\ z \le \delta$ with $0 < \delta < \pi$. Also, $E_1(z) = O(\ln\, z)$ as $z \to 0$. ∎

Definition 1.14. The sequence $\{f_n(z)\}$ is an *asymptotic sequence* for $z \to z_0$, if for each $n = 1, 2, \ldots$, we have $f_{n+1}(z) = o[f_n(z)]$ as $z \to z_0$. ∎

❑ **Example 1.12.** We have

(i) $\{(z - z_0)^n\}$ is an asymptotic sequence for $z \to z_0$.

(ii) If $\{\lambda_n\}$ is a sequence of complex numbers such that $\text{Re}\,\lambda_{n+1} < \text{Re}\,\lambda_n$ for all n, then $\{z^{\lambda_n}\}$ is an asymptotic sequence for $z \to \infty$.

(iii) If $\{\lambda_n\}$ is any sequence of complex numbers, then $\{z^{\lambda_n}\, e^{-nz}\}$ is an asymptotic sequence for $z \to \infty$ in any sector $-\delta \le \arg\, z \le \delta$ with $0 < \delta < \frac{\pi}{2}$. ∎

Definition 1.15. If $\{f_n(z)\}$ is an asymptotic sequence for $z \to z_0$, then

$$f(z) \sim \sum_{n=1}^{N} a_n f_n(z) \tag{1.79}$$

is an *asymptotic expansion* (to N terms) of $f(z)$ as $z \to z_0$ if

$$f(z) - \sum_{n=1}^{N} a_n f_n(z) = o[f_N(z)] \tag{1.80}$$

as $z \to z_0$. The formal series

$$f(z) \sim \sum_{n=1}^{\infty} a_n f_n(z) \tag{1.81}$$

is an *asymptotic series* for $f(z)$ as $z \to z_0$ if

$$f(z) - \sum_{n=1}^{N} a_n f_n(z) = O[f_{N+1}(z)] \tag{1.82}$$

as $z \to z_0$ $(N = 1, 2, \ldots)$. The series (1.82) may converge or diverge, but even if it converges, it need not actually converge to the function, since we say $f(z)$ is *asymptotically equal* to $g(z)$, or $f(z) \sim g(z)$, as $z \to z_0$ with respect to the asymptotic sequence $\{f_n(z)\}$ if

$$f(z) - g(z) = o[f_n(z)] \tag{1.83}$$

as $z \to z_0$ for $n = 1, 2, \ldots$. For example, we have

$$f(z) \sim f(z) + e^{-z} \tag{1.84}$$

with respect to the sequence $\{z^{-n}\}$ as $z \to \infty$ in any sector with $\text{Re}\, z > 0$. Thus a function need not be uniquely determined by its asymptotic series. ∎

Of special interest are asymptotic power series

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} \tag{1.85}$$

for $z \to \infty$ (generally restricted to some sector in the complex plane). Such a series can be integrated term by term, so that if $F'(z) = f(z)$, then

$$F(z) \sim a_0 z + a_1 \ln z + c - \sum_{n=1}^{\infty} \frac{a_{n+1}}{n z^n} \tag{1.86}$$

for $z \to \infty$. On the other hand, the derivative

$$f'(z) \sim -\sum_{n=1}^{\infty} \frac{n a_n}{z^{n+1}} \tag{1.87}$$

only if it is known that $f'(z)$ has an asymptotic power series expansion.

1.4.3 Laplace Integral; Watson's Lemma

Now consider the problem of finding an asymptotic expansion of the integral

$$J(x) = \int_0^a F(t) e^{-xt}\, dt \tag{1.88}$$

for x large and positive (the variable is called x here to emphasize that it is real, although the series derived can often be extended into a sector of the complex plane). It should be clear that such an asymptotic expansion will depend on the behavior of $F(t)$ near $t = 0$, since that is where the exponential factor is the largest, especially in the limit of large positive x. The important result is contained in

Theorem 1.6. *(Watson's lemma).* Suppose that the function $F(t)$ in Eq. (1.88) is integrable on $0 \leq x \leq a$, with an asymptotic expansion for $t \to 0+$ of the form

$$F(t) \sim t^b \sum_{n=0}^{\infty} c_n t^n \tag{1.89}$$

with $b > -1$. Then an asymptotic expansion for $J(x)$ as $x \to \infty$ is

$$J(x) \sim \sum_{n=0}^{\infty} c_n \frac{\Gamma(n+b+1)}{x^{n+b+1}} \tag{1.90}$$

Here

$$\Gamma(\xi + 1) \equiv \int_0^{\infty} t^{\xi} e^{-t}\, dt \tag{1.91}$$

is the Γ-function, which will be discussed at length in Chapter 4. Note that for integer values of the argument, we have $\Gamma(n+1) = n!$.

Proof. To derive the series (1.90), let $0 < \varepsilon < a$, and consider the integral

$$J_\varepsilon(x) \equiv \int_0^{\varepsilon} F(t) e^{-xt}\, dt \sim \sum_{n=0}^{\infty} c_n \int_0^{\varepsilon} t^{n+b}\, e^{-xt}\, dt \tag{1.92}$$

Note that

$$J(x) - J_\varepsilon(x) = \int_\varepsilon^a F(t)e^{-xt}\,dt = e^{-\varepsilon\,x}\int_0^{a-\varepsilon} F(\tau+\varepsilon)e^{-x\tau}\,d\tau \tag{1.93}$$

is exponentially small compared to $J(x)$ for $x \to \infty$, since $F(t)$ is assumed to be integrable on $0 \le t \le a$. Hence $J(x)$ and $J_\varepsilon(x)$ are approximated by the same asymptotic power series.

The asymptotic character of the series (1.89) implies that for any N, we can choose ε small enough that the error term

$$\Delta_\varepsilon^N(x) \equiv \left| J_\varepsilon(x) - \sum_{n=0}^{N-1} c_n \int_0^\varepsilon t^{n+b}e^{-xt}\,dt \right| < C\int_0^\varepsilon t^{N+b}e^{-xt}\,dt \tag{1.94}$$

for some constant C. But we also know that

$$\begin{aligned}\frac{\Gamma(N+b+1)}{x^{N+b+1}} - \int_0^\varepsilon t^{n+b}e^{-xt}\,dt &= \int_\varepsilon^\infty t^{n+b}e^{-xt}\,dt \\ &= e^{-\varepsilon x}\int_0^\infty (\tau+\varepsilon)^{n+b}e^{-x\tau}\,d\tau\end{aligned} \tag{1.95}$$

The right-hand side is exponentially small for $x \to \infty$; hence the error term is bounded by

$$\left|\Delta_\varepsilon^N(x)\right| < C\,\frac{\Gamma(N+b+1)}{x^{N+b+1}} \tag{1.96}$$

Thus the series on the right-hand side of Eq. (1.90) is an asymptotic power series for $J_\varepsilon(x)$ and thus also for $J(x)$. ■

We can use Watson's lemma to derive an asymptotic expansion for $z \to \infty$ of a function $I(z)$ defined by the integral representation (*Laplace integral*)

$$I(z) = \int_0^a f(t)e^{zh(t)}\,dt \tag{1.97}$$

with $f(t)$ and $h(t)$ continuous real functions[3] on the interval $0 \le t \le a$. For z large and positive, we can expect that the most important contribution to the integral will be from the region in t near the maximum of $h(t)$, with contributions from outside this region being exponentially small in the limit $\mathrm{Re}\,z \to +\infty$. There is actually no loss of generality in assuming that the maximum of $h(t)$ occurs at $t = 0$.[4]

The integral (1.97) can be converted to the form (1.88) by introducing a new variable $u \equiv h(0) - h(t)$, and approximating $I(z)$ by

$$I(z) \sim I_\varepsilon(z) \equiv -e^{zh(0)}\int_0^\varepsilon \frac{f(t)}{h'(t)}\,e^{-zu}\,du \tag{1.98}$$

[3]It is enough that $f(t)$ is integrable, but we will rarely be concerned about making the most general technical assumptions.

[4]Suppose $h(t)$ has a maximum at an interior point ($t = b$, say) of the interval of integration. Then we can split the integral (1.97) into two parts, the first an integral from 0 to b, the second an integral from b to a, and apply the present method to each part.

The asymptotic expansion of $I_\varepsilon(z)$ is then obtained from the expansion of $f(t)/h'(t)$ for $t \to 0+$, as just illustrated, provided that such an expansion in the form (1.89) exists. Note that the upper limit $\varepsilon\ (>0)$ in this integral can be chosen at will. This method of generating asymptotic series is due to Laplace.

❑ **Example 1.13.** Consider the integral

$$I(z) = \int_0^\infty e^{-z \sinh t}\, dt \tag{1.99}$$

Changing the variable of integration to $u = \sinh t$ gives

$$I(z) = \int_0^\infty (1+u^2)^{-1/2}\, e^{-zu}\, du \tag{1.100}$$

Expanding

$$(1+u^2)^{-1/2} = \sum_{n=0}^\infty (-1)^n \frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}\, u^{2n} \tag{1.101}$$

then gives the asymptotic series

$$I(z) \sim e^{-z} \sum_{n=0}^\infty (-1)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{(2n)!}{n!} \frac{1}{z^{2n+1}} \tag{1.102}$$

for $z \to \infty$ with $|\arg z| \le \frac{\pi}{2} - \delta$ and fixed $0 \le \delta < \frac{\pi}{2}$. ∎

Now suppose the function $h(t)$ in Eq. (1.97) has a maximum at $t = 0$, with the expansion

$$h(t) \sim h(0) - At^p + \cdots \tag{1.103}$$

for $t \cong 0$, with $A > 0$ and $p > 0$. Then we can introduce a new variable $u = t^p$ into Eq. (1.97), which gives

$$I(z) \sim \frac{1}{p}\, e^{zh(0)}\, f(0) \int_0^\infty u^{\frac{1}{p}}\, e^{-Auz}\, \frac{du}{u} = e^{zh(0)}\, f(0)\, \frac{\Gamma(\frac{1}{p})}{p\,(Az)^{\frac{1}{p}}} \tag{1.104}$$

Note that $\Gamma(\frac{1}{p}) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$ in the important case $p = 2$ that corresponds to the usual quadratic behavior of a function near a maximum. In any case, the leading behavior of $I(z)$ for $z \to \infty$ (with $\mathrm{Re}\, z > 0$) follows directly from the leading behavior of $h(t)$ near $t = 0$.

❑ **Example 1.14.** Consider the integral

$$K_0(z) = \int_0^\infty e^{-z \cosh t}\, dt \tag{1.105}$$

which is a known representation for the modified Bessel function $K_0(z)$. Since $\cosh t \cong 1 + t^2/2$ near $t = 0$, the leading term in the asymptotic expansion of $K_0(z)$ for $z \to \infty$ is given by

$$K_0(z) \sim e^{-z} \sqrt{\frac{\pi}{2z}} \tag{1.106}$$

Here the complete asymptotic expansion of $K_0(z)$ can be derived by changing the variable of integration to $u \equiv \cosh t$. This gives

$$\begin{aligned} K_0(z) &= \int_1^\infty (u^2 - 1)^{-1/2} e^{-zu}\, du \\ &= \sqrt{\tfrac{1}{2}}\, e^{-z} \int_0^\infty v^{-1/2} (1 + \tfrac{1}{2} v)^{-1/2} e^{-zv}\, dv \end{aligned} \tag{1.107}$$

($v = u - 1$). Expanding $(1 + \frac{1}{2}v)^{-1/2}$ then provides the asymptotic series

$$K_0(z) \sim e^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{[\Gamma(n + \frac{1}{2})]^2}{n!\, \Gamma(\frac{1}{2})} \left(\frac{1}{2z}\right)^{n+\frac{1}{2}} \tag{1.108}$$

again for $z \to \infty$ with $|\arg z| \le \frac{\pi}{2} - \delta$ and fixed $0 \le \delta < \frac{\pi}{2}$. ∎

The method introduced here must be further modified if either of the functions $f(t)$ or $h(t)$ in Eq. (1.97) does not have an asymptotic power series expansion for $t \to 0$. Consider, for example, the Γ-function introduced above in Eq. (1.91), which we can write in the form

$$\Gamma(\xi + 1) = \int_0^\infty e^{\xi \ln t}\, e^{-t}\, dt \tag{1.109}$$

The standard method to find an asymptotic expansion for $\xi \to +\infty$ does not work here, since $\ln\, t$ has no power series expansion for $t \to 0$. However, we can note that the argument $(-t + \xi\, \ln\, t)$ of the exponential has a maximum for $t = \xi$. Since the location of the maximum depends on ξ (it is a *moving maximum*), we change variables and let $t \equiv \xi u$, so that Eq. (1.109) becomes

$$\Gamma(\xi + 1) = \xi^{\xi+1} \int_0^\infty e^{\xi \ln u}\, e^{-\xi u}\, du \tag{1.110}$$

Now the argument in the exponent can be expanded about the maximum at $u = 1$ to give

$$\Gamma(\xi + 1) \cong \xi^{\xi+1} e^{-\xi} \int_0^\infty e^{-\frac{1}{2}\xi(u-1)^2}\, du \cong \sqrt{2\pi\xi} \left(\frac{\xi}{e}\right)^{\xi} \tag{1.111}$$

This is the first term in *Stirling's expansion* of the Γ-function; the remaining terms will be derived in Chapter 4.

A Iterated Maps, Period Doubling, and Chaos

We have been concerned in this chapter with the properties of infinite sequences and series from the point of view of classical mathematics, in which the important question is whether or not the sequence or series converges, with asymptotic series recognized as useful for characterizing the limiting behavior of functions, and for approximate evaluation of the functions.

Sequences generated by dynamical systems can have a richer structure. For example, the successive intersections of a particle trajectory with a fixed plane through which the trajectory passes more or less periodically, or the population counts of various species in an ecosystem at definite time intervals, can be treated as sequences generated by a map T that takes each of the possible initial states of the system into its successor. The qualitative properties of such maps are interesting and varied.

As a simple prototype of such a map, consider the *logistic map*

$$T_\lambda : \; x \mapsto f_\lambda(x) \equiv \lambda x(1-x) \tag{1.A1}$$

that maps the unit interval $0 < x < 1$ into itself for $0 < \lambda < 4$ (the maximum value of $x(1-x)$ in the unit interval is $1/4$). Starting from a generic point x_0 in the unit interval, T_λ generates a sequence $\{x_n\}$ defined by

$$x_{n+1} = \lambda \, x_n \, (1 - x_n) \tag{1.A2}$$

If $\lambda < 1$, we have

$$x_{n+1} < \lambda \, x_n < \cdots < \lambda^{n+1} x_0 < \lambda^{n+1} \tag{1.A3}$$

and the sequence converges to 0. But the sequence does not converge to 0 if $\lambda > 1$, and the behavior of the sequence as λ increases is quite interesting.

Remark. A generic map of the type (1.A1) that maps a coordinate space (or *manifold*, to be introduced in Chapter 3) into itself, defines a *discrete-time dynamical system* generated by iterations of the map. The bibliography at the end of the chapter has some suggestions for further reading. □

To analyze the behavior of the sequence in general, note that the map (1.A1) has *fixed points* x_* (points for which $x_* = f_\lambda\,(x_*)$) at

$$x_* = 0, 1 - \tfrac{1}{\lambda} \tag{1.A4}$$

If the sequence (1.A2) starts at one of these points, it will remain there, but it is important to know how the sequence develops from an initial value of x near one of the fixed points. If an initial point close to the fixed point is driven toward the fixed point by successive iterations of the map, then the fixed point is *stable*; if it is driven away from the fixed point, then the fixed point is *unstable*. The sequence can only converge, if it converges at all, to one of its fixed points, and indeed only to a stable fixed point.

To determine the stability of the fixed points in Eq. (1.A4), note that

$$x_{n+1} = f_\lambda\,(x_n) \cong x_* + f'_\lambda\,(x_*)\,(x_n - x_*) \tag{1.A5}$$

for $x_n \cong x_*$, so that

$$\varrho_n \equiv \frac{x_{n+1} - x_*}{x_n - x_*} \cong f'_\lambda(x_*) \tag{1.A6}$$

Stability of the fixed point x_* requires $\{x_n\} \to x_*$ from a starting point sufficiently close to x_*. Hence it is necessary that $|\varrho_n| < 1$ for large n, which requires

$$-1 < f'_\lambda(x_*) < 1 \tag{1.A7}$$

This criterion for the stability of the fixed point is quite general. Note that if

$$f'_\lambda(x_*) = 0 \tag{1.A8}$$

the convergence of the sequence will be especially rapid. With $\varepsilon_n = x_n - x_*$, we have

$$\varepsilon_{n+1} \cong \tfrac{1}{2} f''(x_*)\varepsilon_n^2 \tag{1.A9}$$

and the convergence to the fixed point is exponential; the fixed point is *superstable*.

➔ **Exercise 1.A1.** Find the values of λ for which each of the fixed points in (1.A4) is superstable. □

Remark. The case $|f'_\lambda(x_*)| = 1$ requires special attention, since the ratio test fails. The fixed point may be stable in this case as well. □

For the map defined by Eq. (1.A2), we have

$$f'_\lambda(x_*) = \lambda(1 - 2x_*) \tag{1.A10}$$

so the fixed point $x_* = 0$ is stable only for $\lambda \leq 1$, while the fixed point $x_* = 1 - 1/\lambda$ is stable for $1 \leq \lambda \leq 3$. Hence for $1 \leq \lambda \leq 3$, the sequence $\{x_n\}$ converges,

$$\{x_n\} \to 1 - \tfrac{1}{\lambda} \equiv x_\lambda \tag{1.A11}$$

It requires proof that this is true for any initial value x_0 in the interval $(0, 1)$, but a numerical experiment starting from a few randomly chosen points may be convincing.

What happens for $\lambda > 3$? For λ slightly above 3 ($\lambda = 3.1$, say), a numerical study shows that the sequence begins to oscillate between two fixed numbers that vary continuously from $x_* = \frac{2}{3}$ as λ is increased above 3, and bracket the now unstable fixed point $x_* = x_\lambda$. To study this behavior analytically, consider the iterated sequence

$$x_{n+2} = \lambda^2 x_n(1 - x_n)\left[1 - \lambda x_n(1 - x_n)\right] = f(f(x_n)) \equiv f^{[2]}(x_n) \tag{1.A12}$$

This sequence still has the fixed points given by Eq. (1.A4), but two new fixed points

$$x_*^\pm \equiv \frac{1}{2\lambda}\left\{\lambda + 1 \pm \sqrt{(\lambda + 1)(\lambda - 3)}\,\right\} \tag{1.A13}$$

appear. These new fixed points are real for $\lambda > 3$, and the original sequence (1.A2) eventually appears to oscillate between them (with *period* 2) for $\lambda > 3$.

➔ **Exercise 1.A2.** Derive the result (1.A13) for the fixed points of the second iterate $f^{[2]}$ of the map (1.A1) as defined in Eq. (1.A12). Sketch on a graph the behavior of these fixed points, as well as the fixed points (1.A4) of the original map, as a function of λ for $3 \leq \lambda < 4$. Then derive the result (1.A16) for the value of λ at which these fixed points become unstable, leading to a bifurcation of the sequence into a limit cycle of period 4. □

The derivative of the iterated map $f^{[2]}$ is given by

$$f^{[2]\prime}(x) = f'(f(x))\, f'(x) \tag{1.A14}$$

which at the fixed points (1.A13) becomes

$$f^{[2]\prime}(x_*^{\pm}) = f'(x_*^{+})\, f'(x_*^{-}) = 4 + 2\lambda - \lambda^2 \tag{1.A15}$$

Thus $f^{[2]\prime}(x_*^{\pm}) = 1$ at $\lambda = 3$, and decreases to -1 as λ increases from $\lambda = 3$ to

$$\lambda = 1 + \sqrt{6} \cong 3.4495\ldots \tag{1.A16}$$

when the sequence undergoes a second *bifurcation* into a stable cycle of length 4. Successive period doublings continue after shorter and shorter intervals of λ, until at

$$\lambda \cong 3.56994\ldots \equiv \lambda_c \tag{1.A17}$$

the sequence becomes *chaotic*. Iterations of the sequence starting from nearby points become widely separated, and the sequence does not approach any limiting cycle.

This is not quite the whole story, however. In the interval $\lambda_c < \lambda < 4$, there are islands of periodicity, in which the sequence converges to a cycle of period p for a range of λ, followed (as λ increases) by a series of period doublings to cycles of periods $2p$, $4p$, $8p$, ... and eventual reversion to chaos. There is one island associated with period 3 and its doublings, which for the sequence (1.A2) begins at

$$\lambda = 1 + \sqrt{8} \cong 3.828\ldots \tag{1.A18}$$

and one or more islands with each integer as fundamental period together with the sequence of period doublings. In Fig. 1.1, the behavior of the iterates of the map is shown as a function of λ; the first three period doublings, as well as the interval with stable period 3, are clearly visible. For further details, see the book by Devaney cited in the bibliography at the end of the chapter.

The behavior of the iterates of the map (1.A2) as the parameter λ varies is not restricted to the logistic map, but is shared by a wide class of maps of the unit interval $I \equiv (0, 1)$ into itself. Let

$$T_\lambda :\ x \mapsto \lambda\, f(x) \tag{1.A19}$$

be a map $I \to I$ such that $f(x)$ is continuously differentiable and $f'(x)$ is nonincreasing on I, with $f(0) = 0 = f(1)$, $f'(0) > 0$, $f'(1) < 0$. These conditions mean that $f(x)$ is concave downward in the interval I, increasing monotonically from 0 to a single maximum in the interval, and then decreasing monotonically to 0 at the end of the interval.

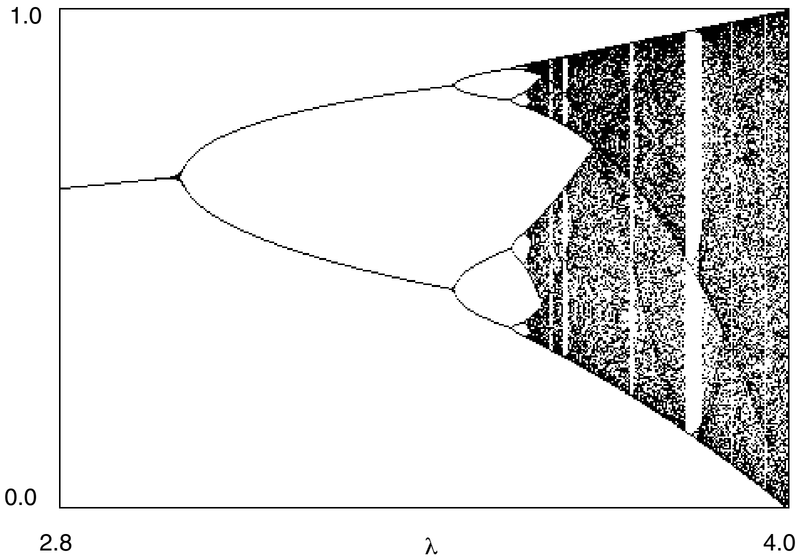

Figure 1.1: Iterates of the map (1.A2) for λ between 2.8 and 4.0. Shown are 100 iterates of the map after first iterating 200 times to let the dependence on the initial point die down.

If $f(x)$ satisfies these conditions, then T_λ shows the same qualitative behavior of period doubling, followed by a chaotic region with islands of periodicity, as a function of λ. Furthermore, if λ_n denotes the value of λ at which the nth period doubling occurs, then

$$\lim_{n\to\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}} \equiv \delta = 4.6692\ \ldots \tag{1.A20}$$

is a *universal* constant, discovered by Feigenbaum in the late 1970s, independent of the further details of the map.

A simple context in which the sequence (1.A2) arises is the model of a biological species whose population in generation $n + 1$ is related to the population in generation n by

$$p_{n+1} \;=\; r\,p_n - a\,p_n^2 \tag{1.A21}$$

Here $r > 0$ corresponds to the natural growth rate ($r > 1$ if the species is not to become extinct), and $a > 0$ corresponds to a natural limitation on the growth of the population (due to finite food supply, for example). Equation (1.A21) implies that the population is limited to

$$p < p_{\max} \equiv r/a \tag{1.A22}$$

and rescaling Eq. (1.A21) by defining $x_n \equiv p_n/p_{\max}$ leads precisely to Eq. (1.A2) with $\lambda = r$. While this model, as well as some related models given in the problems, is oversimplified, period doubling has actually been observed in biological systems. For examples, see Chapter 2 of the book by May cited in the bibliography.

Bibliography and Notes

The first three sections of this chapter are intended mainly as a review of topics that will be familiar to students who have taken a standard advanced calculus course, and no special references are given to textbooks at that level. A classic reference dealing with advanced methods of analysis is

E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (4th edition), Cambridge University Press (1958).

The first edition of this work was published in 1902 but it is valuable even today. In addition to its excellent and thorough treatment of the classical aspects of complex analysis and the theory of special functions, it contains many of the notorious Cambridge Tripos problems, which the modern reader may find even more challenging than the students of the time!

A basic reference on theory of convergence of sequences and series is

Konrad Knopp, *Infinite Sequences and Series*, Dover (1956).

This compact monograph summarizes the useful tests for convergence of series, and gives a collection of elementary examples.

Two books that specialize in the study of asymptotic expansions are

A. Erdélyi, *Asymptotic Expansions*, Dover (1955), and
N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals*, Dover (1986).

The first of these is a concise survey of methods of generating asymptotic expansions, both from integral representations and from differential equations. The second is a more comprehensive survey of the various methods used to generate asymptotic expansions of functions defined by integral representations. It also has many examples of the physical and mathematical contexts in which such integrals may occur. The book by Whittaker and Watson noted above, the book on functions of a complex variable by Carrier, Krook, and Pierson in Chapter 4 and the book on advanced differential equations by Bender and Orszag cited in Chapter 5 also deal with asymptotic expansions.

A readable introduction to the theory of bifurcation and chaos is

R. L. Devaney, *An Introduction to Chaotic Dynamical Systems* (2nd edition), Westview Press (2003).

Starting at the level of a student who has studied ordinary differential equations, this book clearly explains the mathematical foundations of the phenomena that occur in the study of iterated maps. A comprehensive introduction to chaos in discrete and continuous dynamical systems is

Kathleen T. Alligood, Tim D. Sauer, and James A. Yorke, *Chaos: An Introduction to Dynamical Systems*, Springer (1997).

Each chapter has a serious computer project at the end, as well as simpler exercises. A more advanced book is

Edward Ott, *Chaos in Dynamical Systems*, Cambridge University Press (1993).

Two early collections of reprints and review articles on the relevance of these phenomena to physical and biological systems are

R. M. May (ed.), *Theoretical Ecology* (2nd edition), Blackwell Scientific Publishers, Oxford (1981), and

P. Cvitanoviç (ed.), *Universality in Chaos* (2nd edition), Adam Hilger Ltd., Bristol (1989).

The first of these is a collection of specially written articles by May and others on various aspects of theoretical ecology. Of special interest here is the observation of the phenomenon of period doubling in ecosystems. The second book contains a collection of reprints of classic articles by Lorenz, May, Hénon, Feigenbaum, and others, leading to the modern studies of chaos and related behavior of dynamical systems, with some useful introductory notes by Cvitanoviç.

The reader should be aware that the behavior of complex dynamical systems is an important area of ongoing research, so that it is important to look at the current literature to get an up-to-date view of the subject. Nevertheless, the concepts presented here and in later chapters are fundamental to the field. Further readings with greater emphasis on differential equations and partial differential equations are found at the end of Chapters 2 and 8.

Problems[5]

1. Show that

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \sum_{k=0}^{\infty}\frac{z^k}{k!} \ (= e^z)$$

2. Show that

$$\zeta(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-1}$$

where $\prod_p$ denotes a product over all *primes* p.

3. Investigate the convergence of the following series:

(i) $$\sum_{n=1}^{\infty}\left\{\frac{1}{n}-\ln\left(1+\frac{1}{n}\right)\right\}$$

(ii) $$\sum_{n=1}^{\infty}\left\{1-n\,\ln\left(\frac{2n+1}{2n-1}\right)\right\}$$

(iii) $$\sum_{n=2}^{\infty}\frac{1}{n^a\,(\ln n)^b}$$

Explain how convergence depends on the complex numbers a, b in (iii).

[5] When a proposition is simply stated, the problem is to prove it, or to give a counterexample that shows it is false.

4. Investigate the convergence of the following series:

(i) $$\sum_{n=0}^{\infty} (n+1)^a\, z^n$$

(ii) $$\sum_{n=0}^{\infty} \frac{(n+n_1)!\,(n+n_2)!}{n!\,(n+n_3)!}\, z^n$$

(iii) $$\sum_{n=0}^{\infty} e^{-na}\, \cos(bn^2 z)$$

where a, b are real numbers, n_1, n_2, n_3 are positive integers, and z is a (variable) complex number. How do the convergence properties depend on a, b, z ?

5. Find the sums of the following series:

(i) $$S = 1 + \frac{1}{4} - \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \cdots$$

(ii) $$S = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 6} + \cdots$$

(iii) $$S = \frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \cdots$$

(iv) $$f(z) = \sum_{n=0}^{\infty} (-1)^n\, \frac{(n+1)^2}{(2n+1)!}\, z^{2n+1}$$

6. Find a closed form expression for the sums of the series

$$S_p \equiv \sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+p)}$$

$(p = 0, 1, 2, \ldots)$.

Remark. The result obtained here can be used to improve the rate of convergence of a series whose terms tend to zero like $1/n^{p+1}$ for large n; subtracting a suitably chosen multiple of S_p from the series will leave a series whose terms tend to zero at least as fast as $1/n^{p+2}$ for large n. As a further exercise, apply this method to accelerate the convergence of the series for the ζ-function $\zeta(p)$ with p integer. □

7. The quantum states of a simple harmonic oscillator with frequency ν are the states $|n\rangle$ with energies

$$E_n = (n + \tfrac{1}{2})h\nu$$

$(n = 0, 1, 2, \ldots)$ where h is Planck's constant. For an ensemble of such oscillators in thermal equilibrium at temperature T, the probability of finding an oscillator in the state $|n\rangle$ is given by

$$P_n = A \, \exp\left(-\frac{E_n}{kT}\right)$$

where A is a constant to be determined, and the exponential factor is the standard *Boltzmann factor*.

(i) Evaluate the constant A by requiring

$$\sum_0^{\infty} P_n = 1$$

(ii) Find the average energy $\langle E(T)\rangle$ of a single oscillator of the ensemble.

Remark. These results are used in the study of blackbody radiation in Problem 4.9. □

8. Investigate the convergence of the following products:

(i) $$\prod_{m=1}^{\infty} \frac{m(m+a+b)}{(m+a)(m+b)}$$

(ii) $$\prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right)$$

where a, b are the real numbers and z is a (variable) complex number.

9. Evaluate the infinite product

$$\prod_{m=1}^{\infty} \left\{\frac{1 + \exp(i\omega/2^m)}{2}\right\}$$

Hint. Note that $1 + e^{\frac{1}{2}i\omega} = (1 - e^{i\omega})/(1 - e^{\frac{1}{2}i\omega})$.

10. Evaluate the infinite products

(i) $$\prod_{n=1}^{\infty} \left\{1 - \frac{1}{(n+1)^2}\right\}$$

(ii) $$\prod_{n=2}^{\infty} \left(\frac{n^3 - 1}{n^3 + 1}\right)$$

11. Show that

$$\prod_{m=0}^{\infty} \left(1 + z^{2^m}\right) = \frac{1}{1 - z}$$

12. The Euler–Mascheroni constant γ is defined by

$$\gamma \equiv \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{1}{n} - \ln(N+1) \right\}$$

(i) Show that γ is finite (i.e., the limit exists). *Hint.* Show that

$$\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} \left\{ \frac{1}{n} - \ln\left(1 + \frac{1}{n}\right) \right\}$$

(ii) Show that

$$\sum_{n=1}^{N} \frac{1}{n} = \int_0^1 \frac{1-(1-t)^N}{t}\, dt$$

(iii) Show that

$$\int_0^1 \left(\frac{1-e^{-t}}{t} \right) dt - \int_1^\infty \frac{e^{-t}}{t}\, dt = \gamma$$

13. The *error function* $\mathrm{erf}(z)$ is defined by

$$\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$$

(i) Find the power series expansion of $\mathrm{erf}(z)$ about $z = 0$.

(ii) Find an asymptotic expansion of $\mathrm{erf}(z)$ valid for z large and positive. For what range of $\arg z$ is this asymptotic series valid?

(iii) Find an asymptotic expansion of $\mathrm{erf}(z)$ valid for z large and negative. For what range of $\arg z$ is this asymptotic series valid?

14. Find an asymptotic power series expansion of

$$f(z) \equiv \int_0^\infty \frac{e^{-zt}}{1+t^2}\, dt$$

valid for z large and positive. For what range of $\arg z$ is this expansion valid? Give an estimate of the error in truncating the series after N terms.

15. Find an asymptotic power series expansion of

$$f(z) \equiv \int_0^\infty \left(1 + \frac{t}{z}\right)^\alpha e^{-zt}\, dt$$

valid for z large and positive (here α is a complex constant). For what range of $\arg z$ is this expansion valid? Give an estimate of the error in truncating the series after N terms.

16. The modified Bessel function $K_\lambda(z)$ of order λ is defined by the integral representation

$$K_\lambda(z) = \int_0^\infty e^{-z\cosh t}\,\cosh\lambda t\,dt$$

(i) Find an asymptotic expansion of $K_\lambda(z)$ valid for $z \to \infty$ in the right half-plane with λ fixed.

(ii) Find an asymptotic expansion of $K_\lambda(z)$ for $\lambda \to \infty$ with z fixed in the right half of the complex plane (Re $z > 0$).

17. Find an asymptotic expansion of

$$f(z) \equiv \int_0^\infty e^{-zt-1/t}\,dt$$

valid for z large and positive.

18. The reaction rate for the fusion of nuclei A and B in a hot gas (in the center of a star, for example) at temperature T can be expressed as

$$R(T) = \frac{C}{(kT)^{\frac{3}{2}}}\int_0^\infty S(E)\exp\left(-\frac{b}{\sqrt{E}} - \frac{E}{kT}\right)dE \qquad (*)$$

where $S(E)$ is often a smoothly varying function of the relative energy E. The exponential factor $\exp(-E/kT)$ in $(*)$ is the usual Boltzmann factor, while the factor $\exp(-b/\sqrt{E})$ is the probability of tunneling through the energy barrier created by the Coulomb repulsion between the two nuclei. The constant b is given by

$$b = \frac{Z_A Z_B e^2}{\hbar}\sqrt{\frac{2m_A m_B}{m_A + m_B}}$$

where m_A, m_B are the masses, $Z_A e, Z_B e$ the charges of the two nuclei, and $\hbar$ is Planck's constant.

(i) Find the energy $E_* = E_*(T)$ at which the integrand in $(*)$ is a maximum, neglecting the energy dependence of $S(E)$.

(ii) Find the width $\Delta = \Delta(T)$ of the peak of the integrand near E_*, again neglecting the energy dependence of $S(E)$.

(iii) Find an approximate value for $R(T)$, assuming $\Delta \ll E_*$.

Remark. A detailed discussion of nuclear reaction rates in stars can be found in

> C. E. Rolfs and W. S. Rodney, *Cauldrons in the Cosmos: Nuclear Astrophysics*, University of Chicago Press (1988).

among many other books on the physics of stars. □

19. Find the value(s) of λ for which the fixed points of the iterated logistic map (Eq. (1.A12)) are superstable.

20. Consider the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha x_n(1 - x_n^2)$$

Find the fixed points of this sequence, and the ranges of α for which each fixed point is stable. Also find the values of α for which there is a superstable fixed point.

21. Consider the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n\, e^{\lambda(1-x_n)}$$

with $\lambda > 0$ real, and $x_0 > 0$.

(i) For what range of λ is $\{x_n\}$ bounded?

(ii) For what range of λ is $\{x_n\}$ convergent? Find the limit of the sequence, as a function of λ.

(iii) What can you say about the behavior of the sequence for $\lambda > \lambda_0$, where λ_0 is the largest value of λ for which the sequence converges?

(iv) Does the map

$$T_\lambda : x \mapsto x\, e^{\lambda(1-x)}$$

have any fixed point(s)? What can you say about the stability of the fixed point(s) for various values of λ?

22. Consider the sequence $\{x_n\}$ defined by

$$x_{n+1} = \frac{r\, x_n}{(1 + x_n)^b}$$

with $b, r > 0$ real, and $x_0 > 0$.

(i) For what range of b, r is $\{x_n\}$ bounded for all $x > 0$?

(ii) For what range of b, r is $\{x_n\}$ convergent? Find the limit of the sequence (as a function of b, r).

(iii) What can you say about the behavior of the sequence outside the region in the b–r plane for which the sequence converges?

(iv) Does the map

$$T_{r,b} : x \mapsto \frac{r\, x}{(1 + x)^b}$$

have any fixed point(s)? What can you say about the stability of the fixed point(s) for various values of b, r?