JEE MAIN 2014

MATHEMATICS

1. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n \in N\}$, where N is the set of natural numbers, then $X \cup Y$ is equal to

(1) X (2) Y

(3) N (4) $Y - X$

Solution

$X = \{4^n - 3n - 1 : n \in N\}$

$4^n - 3n - 1 = (1 + 3)^n - 3n - 1$

$= \cancel{{}^nC_0 3^0} + \cancel{{}^nC_1 3^1} + {}^nC_2 3^2 + \cdots {}^nC_n 3^n - \cancel{3n} - \cancel{1}$

$= 9({}^nC_2 + \cdots {}^nC_n 3^{n-2}) = 9k$, where k is an integer, when $n \geq 2$

When $n = 1$, $4^n - 3n - 1 = 4 - 3 - 1 = 0 = 9k$

Therefore, X contains multiples of 9.

Now $Y = \{9(n-1) : n \, \varepsilon \, N\} = \{0, 9, 18, \ldots\}$ = all multiples of 9

Thus, $X \cup Y = Y$

Hence, the correct option is (2).

2. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$

(1) is strictly greater $\frac{5}{2}$

(2) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

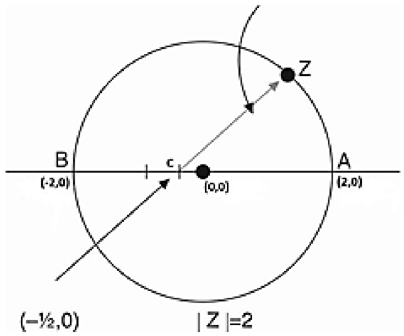
(3) is equal to $\frac{5}{2}$

(4) lies in the interval (1, 2)

Solution

$\text{Min}\left|z + \frac{1}{2}\right|$ is when z is at B.

$$\left|z-\left(-\frac{1}{2}\right)\right|=\left|z+\frac{1}{2}\right|$$



Therefore, $\text{Min}\left|z+\frac{1}{2}\right|=|BC|=1+\frac{1}{2}=\frac{3}{2}$.

Hence $\text{Min}\left|z+\frac{1}{2}\right|$ lies in the interval (1, 2).

Hence, the correct option is (4).

3. If $a \in R$ and the equation, $-3(x-[x])^2+2(x-[x])+a^2=0$ (where $[x]$ denotes the greatest integer $\le x$) has no integral solution, then all possible values of a lie in the interval:

(1) $(-2, -1)$ (2) $(-\infty, -2) \cup (2, \infty)$

(3) $(-1, 0) \cup (0, 1)$ (4) $(1, 2)$

Solution

$-3(x-[x])^2+2(x-[x])+a^2=0 \Rightarrow -3\{x\}^2+2\{x\}+a^2=0$

$$\Rightarrow a^2=3\{x\}^2-2\{x\}=3\left[\{x\}^2-\frac{2}{3}\{x\}+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2\right]=3\left[\{x\}-\frac{1}{3}\right]^2-\frac{1}{3}$$

Now we know that $0 \le \{x\} < 1$.

Therefore, $-\frac{1}{3}\le\{x\}-\frac{1}{3}<\frac{2}{3}\Rightarrow 0\le\left[\{x\}-\frac{1}{3}\right]^2<\frac{4}{9}\Rightarrow 0\le 3\left[\{x\}-\frac{1}{3}\right]^2<\frac{4}{3}$

$\Rightarrow -\frac{1}{3}\le 3\left[\{x\}-\frac{1}{3}\right]^2-\frac{1}{3}<1$

$\Rightarrow -1 \le a^2 < 1$

Only possibility for non-integral solution is $0 < a^2 < 1$

Thus, $a^2 \ge 0$, but when $a = 0$, there is integral solution for $\{x\} = 0$

Therefore, $(-1, 0) \cup (0, 1)$

Hence, the correct option is (3).

4. Let α and β be the roots of the equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q, r are in A.P. and $\frac{1}{\alpha}+\frac{1}{\beta}=4,$ then the value of $|\alpha-\beta|$ is

(1) $\frac{\sqrt{34}}{9}$ (2) $\frac{2\sqrt{13}}{9}$

(3) $\frac{\sqrt{61}}{9}$ (3) $\frac{2\sqrt{17}}{9}$

Solution

Given that,

$\frac{1}{\alpha}+\frac{1}{\beta}=4$ (1)

$2q = p + r$ (2)

Since α and β are roots of equation $px^2 + qx + r = 0$, we have:

$\alpha+\beta=-\frac{q}{p}$ (3)

$\alpha\beta=\frac{r}{p}$ (4)

From eq (1), we have $\frac{\alpha+\beta}{\alpha\beta}=4$

From eq (2) and (3), we have $-2(\alpha+\beta)=\frac{2q}{p}=\frac{p+r}{p}=1+\frac{r}{p}=1+\alpha\beta$

Therefore, $-2\frac{(\alpha+\beta)}{\alpha\beta}=\frac{1}{\alpha\beta}+1 \Rightarrow -2\times 4=\frac{1}{\alpha\beta}+1 \Rightarrow \frac{1}{\alpha\beta}=-9$

Therefore, equation having roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is $x^2 - 4x - 9 = 0$

Therefore, equation having roots α and β is $\frac{1}{x^2}-\frac{4}{x}-9=0 \Rightarrow 9x^2 + 4x - 1 = 0$

Therefore, $|\alpha-\beta|=\sqrt{|\alpha+\beta|^2-4\alpha\beta}=\sqrt{\frac{16}{81}-4\left(\frac{-1}{9}\right)}=\sqrt{\frac{16+36}{81}}=\frac{\sqrt{52}}{9}=\frac{2\sqrt{13}}{9}$

Hence, the correct option is (2).

5. If $\alpha, \beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then K is equal to

(1) 1 (2) –1

(3) $\alpha\beta$ (4) $\frac{1}{\alpha\beta}$

Solution

$$\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} = \{(1-\alpha)(1-\beta)(\alpha-\beta)^2)\}$$

On comparison with the given equation, we get $K = 1$.

Hence, the correct option is (1).

6. If A is a 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' equals

(1) B^{-1} (2) $(B^{-1})'$

(3) $I + B$ (4) I

Solution

$$BB' = A^{-1}A'(A^{-1}A')' = A^{-1}A'A(A^{-1})' = A^{-1}AA'(A^{-1})' = IA'(A^{-1})' = I(A^{-1}A)' = I.I' = I^2 = I$$

Hence, the correct option is (4).

7. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to

(1) $\left(14, \frac{272}{3}\right)$ (2) $\left(16, \frac{272}{3}\right)$

(3) $\left(16, \frac{251}{3}\right)$ (3) $\left(14, \frac{251}{3}\right)$

Solution

$$(1 + \alpha x + bx^2)(1 - 2x)^{18}$$

$$= (1+ax+bx^2)\{{}^{18}C_0(-2x)^0 + {}^{18}C_1(-2x)^1 + {}^{18}C_2(-2x)^2 + {}^{18}C_3(-2x)^3 + {}^{18}C_4(-2x)^4 + \cdots\}$$

Coefficient of $x^3 = {}^{18}C_3(-2)^3 + a \times {}^{18}C_2(-2)^2 + b \times {}^{18}C_1(-2)$

$$= -\frac{8 \times \overset{3}{\cancel{18}} \times 17 \times 16}{\cancel{3} \times 2 \times 1} + a \times \overset{2}{\cancel{4}} \times \frac{18 \times 17}{\cancel{2} \times 1} - 2b \times 18$$

$$= -4\{2 \times 3 \times 17 \times 16 - 9 \times 17a + ab\} = 0$$

$\Rightarrow 153a - ab = 1632 \Rightarrow 51a - 3b = 544$

Only $\left(16, \frac{272}{3}\right)$ satisfies it

Hence, the correct option is (2).

8. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + 10(11)^9 = k(10)^9$, then k is equal to

(1) 100 (2) 110

(3) $\frac{121}{10}$ (4) $\frac{441}{100}$

Solution

Let $S = 1 \times 10^9 + 2 \times 11^1 \times 10^8 + 3 \times 11^2 \times 10^7 + \ldots 9 \times 10 \times 11^8 + 10 \times 11^9$ (1)

Therefore, $\frac{11}{10}S = 11 \times 10^8 + 2 \times 11^2 \times 10^7 + 3 \times 11^3 \times 10^6 + \ldots 9 \times 11^9$ (2)

On subtracting (2) from (1), we get

$$\frac{1}{10}S = -11 \times 10^8 - 11^2 \times 10^7 - \ldots 11^9 + 11^{10} - 10^9$$

$$= -10^9 - 11 \times 10^8 - 11^2 \times 10^7 - \ldots 11^9 + 11^{10}$$

$$= -10^9 \frac{\left\{(11/10)^{10} - 1\right\}}{(11/10) - 1} + 11^{10} = \frac{-10^9 \left\{(11/10)^{10} - 1\right\}}{1/10} + 11^{10}$$

$$\Rightarrow \frac{1}{10}S = -10^{10}\left\{\frac{11^{10}}{10^{10}} - 1\right\} + 11^{10} = -\cancel{11^{10}} + 10^{10} + \cancel{11^{10}}$$

$S = 10^{11} = 100 \times 10^9$

Therefore, $k = 100$

Hence, the correct option is (1).

9. Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. Then the common ratio of the G.P. is

(1) $2-\sqrt{3}$ (2) $2+\sqrt{3}$

(3) $\sqrt{2}+\sqrt{3}$ (3) $3+\sqrt{2}$

Solution

Let the numbers are a, $a\,r$, $a\,r^2$.

Now $2[2ar] = a + a\,r^2 \Rightarrow 4\,a\,r = a + a\,r^2 \Rightarrow r^2 - 4\,r + 1 = 0$

$$\Rightarrow r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Since the numbers form an increasing GP so $r > 1$.

Therefore, $r = 2+\sqrt{3}$

Hence, the correct option is (2).

10. $\lim\limits_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to

(1) $-\pi$ (2) π

(3) $\frac{\pi}{2}$ (4) 1

Solution

$$\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2} = \frac{\sin\left(\pi(1-\sin^2 x)\right)}{x^2} = \frac{\sin(\pi - \pi \sin^2 x)}{\underset{x \to 0}{x^2}}$$

$$= \frac{\sin(\pi \sin^2 x)}{\underset{x \to 0}{x^2}} = \frac{\sin(\pi \sin^2 x)}{\underset{x \to 0}{\pi \sin^2 x}} \times \frac{\pi \sin^2 x}{x^2}$$

$= 1 \times \pi = \pi$

Hence, the correct option is (2).

11. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to

(1) $\frac{1}{1+\{g(x)\}^5}$ (2) $1+\{g(x)\}^5$

(3) $1 + x^5$ (4) $5x^4$

Solution

Given $g(x) = f^{-1}(x)$

Therefore, $f(g(x)) = x$

$f'(g(x))\, g'(x) = 1$

Thus, $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\frac{1}{1+\{g(x)\}^5}}$

Thus, $g'(x) = 1 + \{g(x)\}^5$

Hence, the correct option is (2).

12. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$

(1) $f'(c) = g'(c)$ (2) $f'(c) = 2g'(c)$

(3) $2f'(c) = g'(c)$ (4) $2f'(c) = 3g'(c)$

Solution

Let $h(x) = f(x) - 2g(x)$

So that $h(0) = f(x) - 2\, g(x) = 2 - 2 \times 0 = 2$

And $h(1) = f(1) - 2\, g(1) = 6 - 2 \times 2 = 2$

Now $h(x)$ is a differentiable function in [0, 1] and $h(0) = h(1)$, so by Rolle's theorem $h'(c) = 0$ for some $c \in (0, 1)$

Therefore, $0 = f'(c) - 2\, g'(c) \Rightarrow f'(c) = 2\, g'(c)$

Hence, the correct option is (2).

13. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log |x| + \beta x^2 + x$, then

(1) $\alpha = 2, \beta = -\frac{1}{2}$ (2) $\alpha = 2, \beta = \frac{1}{2}$

(3) $\alpha = -6, \beta = \frac{1}{2}$ (4) $\alpha = -6, \beta = -\frac{1}{2}$

Solution

$f(x) = \alpha \log(x) + \beta x^2 + x$

$f'(x) = \frac{\alpha}{x} + 2\beta x + 1$

Now $f'(-1) = -\alpha - 2\beta + 1 = 0$

$$f'(2) = \frac{\alpha}{2} + 4\beta x + 1 = 0$$

Thus, the equations are

$$\alpha + 2\beta - 1 = 0 \text{ and } \alpha + 8\beta + 2 = 0$$

On Solving the above equations, we get

$$\beta = -\frac{1}{2}, \alpha = 2$$

Hence, the correct option is (1).

14. The integral $\int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\, dx$ is equal to

(1) $(x+1)e^{x+\frac{1}{x}} + c$

(2) $-xe^{x+\frac{1}{x}} + c$

(3) $(x-1)e^{x+\frac{1}{x}} + c$

(4) $xe^{x+\frac{1}{x}} + c$

Solution

$$\int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx = \int \left\{ e^{x+\frac{1}{x}} dx + x\left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}} \right\} dx = xe^{x+\frac{1}{x}} + c$$

$$\left[\because \int \{xf'(x) + f(x)\} dx = xf(x) + c\right]$$

Hence, the correct option is (4).

15. The integral $\int_0^{\pi} \sqrt{1 + 4\sin^2 \frac{x}{2} - 4\sin\frac{x}{2}}\, dx$ equals

(1) $4\sqrt{3} - 4$

(2) $4\sqrt{3} - 4 - \frac{\pi}{3}$

(3) $\pi - 4$

(3) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$

Solution

$$\int_0^{\pi} \sqrt{1 + 4\sin^2 \frac{x}{2} - 4\sin\frac{x}{2}}\, dx = \int_0^{\pi} \sqrt{\left(1 - 2\sin\frac{x}{2}\right)^2}$$

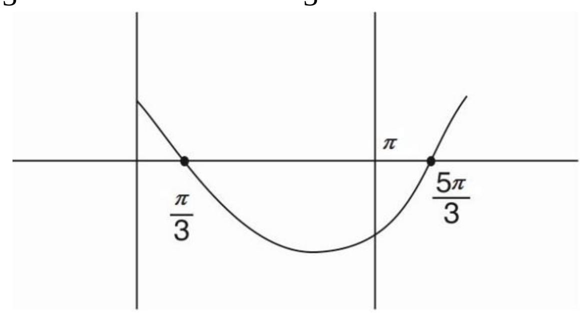
$$= \int_0^{\pi} \left(1 - 2\sin\frac{x}{2}\right) dx = \int_0^{\frac{\pi}{3}} \left(1 - 2\sin\frac{x}{2}\right) dx + \int_{\frac{\pi}{3}}^{2\pi} - \left(1 - 2\sin\frac{x}{2}\right) dx$$

$$=\left[x+\frac{2\cos\frac{x}{2}}{(1/2)}\right]_0^{\frac{\pi}{3}}-\left[x+\frac{2\cos\frac{x}{2}}{(1/2)}\right]_{\frac{\pi}{3}}^{\pi}$$

$$=\left(\frac{\pi}{3}+2\sqrt{3}\right)-(4)-\left[\left(\pi+4(0)\right)-\left(\frac{\pi}{3}+2\sqrt{3}\right)\right]$$

$$=\frac{\pi}{3}+2\sqrt{3}-4-\pi+\frac{\pi}{3}+2\sqrt{3}$$

$$=\frac{2\pi}{3}-\pi+4\sqrt{3}-4=4\sqrt{3}-\frac{\pi}{3}-4$$



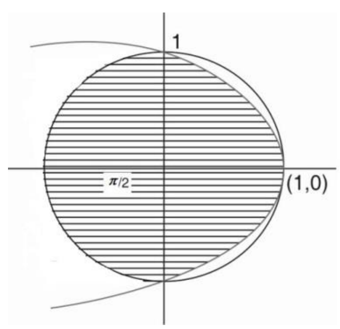
Hence, the correct option is (2).

16. The area of the region described by $A = \{(x, y) : x^2 + y^2 \leq 1 \text{ and } y^2 \leq 1 - x\}$ is

(1) $\frac{\pi}{2}-\frac{2}{3}$ (2) $\frac{\pi}{2}+\frac{2}{3}$

(3) $\frac{\pi}{2}+\frac{4}{3}$ (4) $\frac{\pi}{2}-\frac{4}{3}$

Solution



$$y^2 = 1 - x \Rightarrow x = 1 - y^2$$

Required Area is $=\frac{1}{2}(\pi\times 1^2)+2\int_0^1(1-y^2)dy\ \frac{\pi}{2}+2\left[y-\frac{y^3}{3}\right]_0^1=\frac{\pi}{2}+2\left[\left(1-\frac{1}{3}\right)-0\right]=\frac{\pi}{2}+\frac{4}{3}$

Hence, the correct option is (3).

17. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$ .If $p(0) = 100$, then $p(t)$ equals

(1) $600 - 500\, e^{t/2}$ (2) $400 - 300\, e^{-t/2}$

(3) $400 - 300\, e^{t/2}$ (4) $300 - 200\, e^{-t/2}$

Solution

$$\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200 \Rightarrow \frac{dp(t)}{\frac{1}{2}p(t) - 200} = dt$$

On integrating we get

$$\frac{\log\left|\frac{p(t)}{2} - 200\right|}{(1/2)} = t + c_1 \Rightarrow \frac{\log|p(t) - 400|}{2} = \frac{1}{2}t + \frac{c_1}{2} \qquad (1)$$

Using initial Conditions

$$\frac{\log|100 - 400|}{2} = \frac{c_1}{2} \Rightarrow 2\log 150 = c_1 \qquad (2)$$

From eqn. (1) and eqn(2) we get

$$\log\left|\frac{p(t) - 400}{2}\right| = \frac{1}{2}t + \log 150 \Rightarrow \log\left|\frac{p(t) - 400}{300}\right| = \frac{1}{2}t$$

$$\left|\frac{p(t) - 400}{300}\right| = e^{t/2} \text{or} \left|p(t) - 400\right| = 300e^{t/2}$$

Since initially number of rabbits is 100 and is decreasing

$$-(p(t) - 400) = 300^{t/2} \text{ or } p(t) = 400 - 300e^{t/2}$$

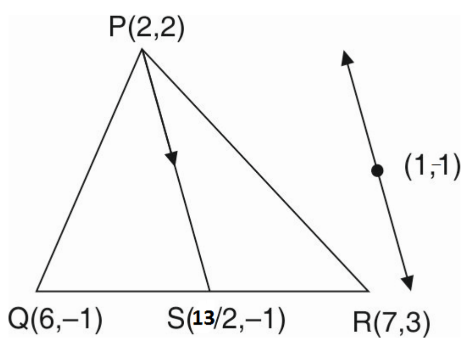
Hence, the correct option is (3).

18. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is

(1) $4x + 7y + 3 = 0$ (2) $2x - 9y - 11 = 0$

(3) $4x - 7y - 11 = 0$ (4) $2x + 9y + 7 = 0$

Solution



Slope of required line is $\frac{1}{(-9/2)} = -\frac{2}{9}$

Equation is $y + 1 = -\frac{2}{9}(x - 1) \Rightarrow 9\,y + 9 = -\,2\,x + 2$ or $2\,x + 9\,y + 7 = 0$

Hence, the correct option is (4).

19. Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes then

(1) $3bc - 2ad = 0$ (2) $3bc + 2ad = 0$

(3) $2bc - 3ad = 0$ (4) $2bc + 3ad = 0$

Solution

$$4ax + 2ay + c = 0$$

$$5bx + 2by + d = 0$$

$$\frac{x}{2ad - abc} = \frac{y}{5bc - 4ad} = \frac{1}{8ab - 10ab}$$

$$x = \frac{\not{2}(ad - bc)}{-\not{2}(ab)} = \frac{bc - ad}{ab}$$

$$y = \frac{5bc - 4ad}{-2ab}$$

Now according to question point being in 4^{th} quadrant and equidistance from axes, lies on $y = -x$

Therefore, $$\frac{5bc - 4ad}{-2\not{ab}} = -\frac{(bc - ad)}{\not{ab}}$$

$5bc - 4ad = 2(bc - ad) \Rightarrow 5bc - 4ad = 2bc - 2ad$

$\Rightarrow 3bc - 2ad = 0$

Hence, the correct option is (1).

20. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is

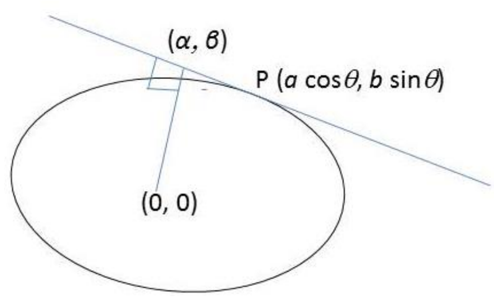
(1) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (2) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

(3) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (4) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

Solution

Ellipse is $\dfrac{x^2}{(\sqrt{6})^2} + \dfrac{y^2}{(\sqrt{2})^2} = 1$

Eqn. of tangent at P is $\dfrac{x\cos\theta}{\sqrt{6}} + \dfrac{y\sin\theta}{\sqrt{2}} = 1$ (1)



Equation of tangent at (α, β) is

$y - \beta = -\dfrac{\alpha}{\beta}(x - \alpha)$ or $\beta y - \beta^2 = -\alpha x + \alpha^2$

or $\alpha x + \beta y = \alpha^2 + \beta^2$ (2)

Comparing Eq (1) and Eq (2), we get

$$\frac{\cos\theta}{\sqrt{6}\alpha} = \frac{\sin\theta}{\sqrt{2}\beta} = \frac{1}{\alpha^2 + \beta^2}$$

or $\sqrt{\dfrac{\cos^2\theta+\sin^2\theta}{6\alpha^2+2\beta^2}} \Rightarrow \dfrac{1}{\sqrt{6\alpha^2+2\beta^2}} = \dfrac{1}{\alpha^2+\beta^2}$

$\Rightarrow 6\alpha^2 + 2\beta^2 = (\alpha^2+\beta^2)$

Therefore, locus is $(x^2+y^2)^2 = 6x^2 + 2y^2$

Hence, the correct option is (1).

21. Let C be the circle with centre at (1, 1) and radius = 1. If T is the circle centred at (0, y), passing through origin and touching the circle C externally, then the radius of T is equal to

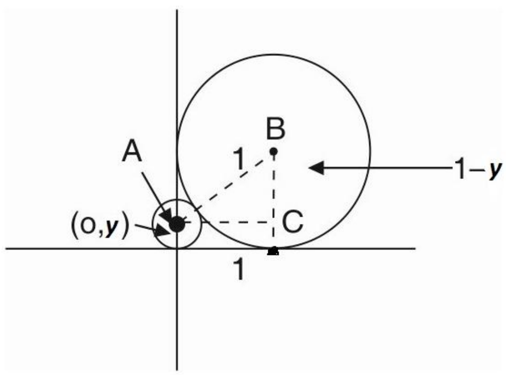
(1) $\dfrac{1}{2}$ (2) $\dfrac{1}{4}$

(3) $\dfrac{\sqrt{3}}{\sqrt{2}}$ (4) $\dfrac{\sqrt{3}}{2}$

Solution

$AB = 1 + y$

$BC = 1 - y$



Now by Pythagoras theorem $(1+y)^2 = 1^2 + (1-y)^2 \Rightarrow 4y = 1$

Therefore, $y = \dfrac{1}{4}$

Hence, the correct option is (2).

22. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is

(1) $\dfrac{1}{8}$ (2) $\dfrac{2}{3}$

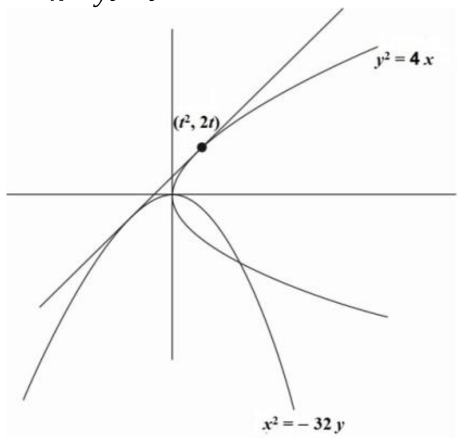
(3) $\dfrac{1}{2}$ (4) $\dfrac{3}{2}$

Solution

Let a Parametric point on $y^2 = 4x$ be $(t^2, 2t)$

Therefore, Equation of tangent is $yt = x + t^2$ (1)

or $x = yt - t^2$



Now solving

$x = yt - t^2$ and $x^2 = -32y$ for points of intersection.

Therefore, $(yt - t^2)^2 = -32y$

or $y^2t^2 + t^4 - 2yt^3 = -32y$

or $y^2t^2 + y(32 - 2t^3) + t^4 = 0$

For line to be tangent, discriminate = 0

Therefore, $(32 - 3t^3)^2 - 4t^6 = 0$ or $4(16 - t^3)^2 - 4(t^3)^2 = 0$

$\Rightarrow (16 - t^3 + t^3)(16 - t^3 - t^3) = 0 \Rightarrow d^3 = 8 \Rightarrow t = 2$

Now slope $\frac{1}{t} = \frac{1}{2}$ (from 1)

Hence, the correct option is (3).

23. The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ in the plane $2x - y + z + 3 = 0$ is the line

(1) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$ (2) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$

(3) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$ (4) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z+2}{5}$

Solution

Since, $3(2) + 1(-1) + (-5)(1) = 6 - 1 - 5 = 0$

Therefore, Line is parallel to plane

Now finding image of point (1, 3, 4) in the plane. Let the image is (x_1, y_1, z_1)

Therefore, $\frac{x_1 - 1}{2} = \frac{y_1 - 3}{-1} = \frac{z_1 - 4}{1} = \frac{-2(2(1) - (3) + (4) + 3)}{2^2 + 1^2 + 1^2} = \frac{-2 \times \not{6}}{\not{6}}$

Therefore, $x_1 = -3, y_1 = 5, z_1 = 2$

Therefore, required image line is $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$

Hence, the correct option is (3).

24. The angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 = m^2 + n^2$ is

(1) $\dfrac{\pi}{6}$ (2) $\dfrac{\pi}{2}$

(3) $\dfrac{\pi}{3}$ (4) $\dfrac{\pi}{4}$

Solution

Since, $l = -m - n$

Therefore, $(-m-n)^2 = m^2 + n^2$ or $2mn = 0 \Rightarrow m = 0$ or $n = 0$

If $m = 0$, then $l^2 = n^2 l = n$

Therefore, $n^2 + 0^2 + n^2 = 1 \Rightarrow n^2 = \dfrac{1}{2} \Rightarrow n = \pm\dfrac{1}{\sqrt{2}}$

Therefore, directions are $\left(\dfrac{-1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}}\right)$

If $n = 0$, then $l = -m \Rightarrow m^2 + m^2 + 0 = 1$

Thus, $m^2 = \dfrac{1}{2} \Rightarrow m = \pm\dfrac{1}{\sqrt{2}}$

Therefore, directions are $\left(\dfrac{-1}{\sqrt{2}}, +\dfrac{1}{\sqrt{2}}, 0\right)$

$$\cos\theta = \frac{1}{2} + 0 + 0 \Rightarrow \theta = \frac{\pi}{3}$$

Hence, the correct option is (3).

25. If $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = \lambda[\vec{a}\ b\ \vec{c}]^2$, then λ is equal to

(1) 0 (2) 1

(3) 2 (3) 3

Solution

$$[\vec{a}\times\vec{b}.\vec{b}\times\vec{c}\vec{c}\times\vec{a}] = (\vec{a}\times\vec{b}).[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})] = (\vec{a}\times\vec{b}).[(\vec{b}\times\vec{c})\times\vec{c}\times\vec{a}]$$

$$= (\vec{a}\times\vec{b}).[(\vec{b}\times\vec{c}\times\vec{a})\vec{c} - (\vec{b}\times\vec{c}\times\vec{c})\vec{a}]$$

$$= (\vec{a}\times\vec{b}).[(\vec{b}\times\vec{c}.\vec{a})\vec{c} - 0]$$

$$= (\vec{a}\times\vec{b}.\vec{c})(\vec{a}\times\vec{b}.\vec{c}) = (\vec{a}\times\vec{b}.\vec{c})^2 = [\vec{a}\vec{b}.\vec{c}]^2$$

Therefore, $\lambda = 1$

Hence, the correct option is (2).

26. Let A and B be two events such that $P(\overline{A\cup B}) = \dfrac{1}{6}$, $P(A\cap B) = \dfrac{1}{4}$ and $P(\overline{A}) = \dfrac{1}{4}$, where $\overline{A}$ stands for the complement of the event A. Then the events A and B are

(1) Independent but not equally likely
(2) Independent and equally likely
(3) Mutually exclusive and independent
(4) Equally likely but not independent

Solution

$$P(\overline{A\cup B})=\frac{1}{C}\Rightarrow P(A\cup B)=1-\frac{1}{6}=\frac{5}{6}$$

$P(A\cap B)=\frac{1}{4}$ (1)

$P(\overline{A})=\frac{1}{4}\Rightarrow P(A)=1-\frac{1}{4}=\frac{3}{4}$

Now, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\frac{5}{6}=\frac{3}{4}+P(B)-\frac{1}{4}$ or $P(B)=\frac{5}{6}-\frac{3}{4}+\frac{1}{4}=\frac{10-9+3}{12}=\frac{4}{12}=\frac{1}{3}$

$P(A)\times P(B)=\frac{3}{4}\times\frac{1}{3}=\frac{1}{4}$ (2)

Therefore, from (1) and (2), A and B are independent.

Since $P(A)\neq P(B)$, therefore not equally likely

Hence, the correct option is (1).

27. The variance of first 50 even natural numbers is

(1) 437 (2) $\frac{437}{4}$

(3) $\frac{833}{4}$ (4) 833

Solution

Variance = σ^2 Mean of squares – Square of Mean

$$=\frac{\sum_{r=1}^{50}(2r)^2}{n}-\left(\frac{\sum_{r=1}^{50}(2r)}{n}\right)^2$$

$$=\frac{4(1^2+2^2+3^2+\cdots 50^2)}{50}-\left\{\frac{2(1+2+3+\cdots 50)}{50}\right\}^2$$

$$=4\times\frac{50(51)(101)}{6\times 50}-4\left(\frac{50\times 51}{2\times 50}\right)^2$$

$= 3434 - 51^2 = 833$

Hence, the correct option is (4).

28. Let $f_k(x)=\frac{1}{k}(\sin^k x+\cos^k x)$ where $x\in\mathbb{R}$ and $k\geq 1$. Then, $f_4(x)-f_6(x)$ equals

(1) $\frac{1}{4}$ (2) $\frac{1}{12}$

(3) $\frac{1}{6}$ (4) $\frac{1}{3}$

Solution

$$f_4(x)-f_6(x)=\frac{1}{4}(\sin^4 x+\cos^4 x)-\frac{1}{6}(\cos^6 x+\sin^6 x)$$

$$= \frac{1}{4}[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x] - \frac{1}{6}[(\cos^2 x + \sin^2 x)^3 - 3\cos^2 x\sin^2 x(\cot^2 x + \sin^2 x)]$$

$$= \frac{1}{4}[1 - 2\sin^2 x\cos^2 x] - \frac{1}{6}[1 - 3\cos^2 x\sin^2 x]$$

$$= \frac{1}{4} - \frac{1}{2}\sin^2 x\cos^2 x - \frac{1}{6} + \frac{1}{2}\cos^2 x\sin^2 x = \frac{3-12}{12} = \frac{1}{12}$$

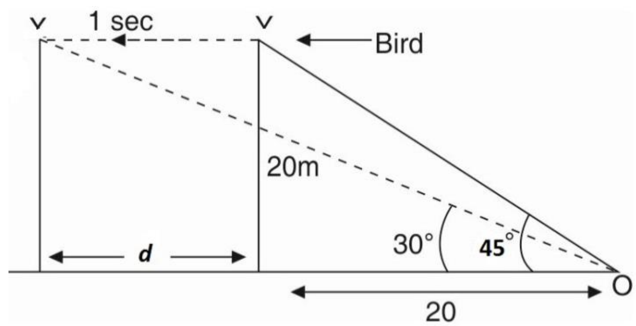
Hence, the correct option is (2).

29. A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45°. It flies off horizontally straight away from the point O. After one second, the elevation of the bird from O is reduced to 30°. Then the speed (in m/s) of the bird is

(1) $20\sqrt{2}$ (2) $20(\sqrt{3}-1)$

(3) $40(\sqrt{2}-1)$ (4) $40(\sqrt{3}-\sqrt{2})$

Solution



$$\frac{d+20}{20} = \cot 30^\circ = \sqrt{3} \Rightarrow d = 20\sqrt{3} - 20 = 20(\sqrt{3}-1)$$

Hence, the correct option is (2).

30. The statement $\sim(p \leftrightarrow \sim q)$ is

(1) A tautology (2) A fallacy

(3) Equivalent to $p \leftrightarrow q$ (4) Equivalent to $\sim p \leftrightarrow q$

Solution

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$	$p \leftrightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

↑ Bi-conditional

Hence, the correct option is (3).