# JEE MAIN 2015

## MATHEMATICS

**1.** The sum of coefficients of integral powers of $x$ in the binomial expansion of $\left(1-2\sqrt{x}\right)^{50}$ is:

(1) $\frac{1}{2}(3^{50})$

(2) $\frac{1}{2}(3^{50}-1)$

(3) $\frac{1}{2}(2^{50}+1)$

(4) $\frac{1}{2}(3^{50}+1)$

**Solution**

$$(1-2\sqrt{x})^{50}=\sum_{r=0}^{50} {}^{50}C_r(1)^{50-r}(-2\sqrt{x})^2$$

$$=\sum_{r=0}^{50} {}^{50}C_r(-2)^r(x)^{r/2}$$

Therefore, for integer powers of $x$, $r \in <0, 2, 4, 6, \ldots, 50$

So the required sum of coefficients = ${}^{50}C_0 + {}^{50}C_2(2)^2 + {}^{50}C_4(2)^4 + \cdots + {}^{50}C_{50}(2)^{50}$ (1)

Since, $(1+2)^{50}+(1-2)^{50}=2\left[{}^{50}C_0 + {}^{50}C_2(2)^2 + {}^{50}C_4(2)^4 + \cdots + {}^{50}C_{50}(2)^{50}\right]$ (2)

In view of (1) and (2),

Required sum $=\frac{1}{2}[(3)^{50}+1]$

**Hence, the correct option is (2).**

**2.** Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$ If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]=3,$ the $f(2)$ is equal to:

(1) −4

(2) 0

(3) 4

(4) −8

**Solution**

$$f'(x)=a(x-1)(x-2)(x-k)$$

$$\Rightarrow\ f'(x)=a[x^3-(3+k)x^2+(2+3k)x-2k]$$

$$\Rightarrow\ f(x)=a\left[\frac{x^4}{4}-\frac{(3+k)x^3}{3}+\frac{(2+3k)x^2}{2}-2kx\right]+C$$

$$\Rightarrow\ \frac{f(x)}{x^2}=a\left[\frac{x^2}{4}-\frac{(3+k)x}{3}+\frac{(2+3k)}{2}-\frac{2k}{x}\right]+\frac{C}{x^2}$$

$$\lim_{x\to 0}\left(1+\frac{f(x)}{x^2}\right)=\lim_{x\to 0}\left\{1+a\left[\frac{(2+3k)x^2-4kx+2C}{2x^2}\right]\right\}=3 \text{ (given)}$$

$$\Rightarrow \lim_{x\to 0}\left\{\frac{[a(2+3k)+2]x^2-4kax+2Ca}{2x^2}\right\}=3$$

$\Rightarrow$ $Ca = 0$ (1)

and $\lim_{x\to 0}\left\{\frac{[a(2+3k)+2](2x)-4ka}{4x}\right\}=3$

$\Rightarrow$ $ka = 0$ (2)

and $\lim_{x\to 0}\left\{\frac{[(2+3k)+2](2)}{4}\right\}=3 \Rightarrow a(2 + 3k) + 2 = 6$

$\Rightarrow$ $2a + 3ak = 4 \Rightarrow a = 2$ ($\because$ $ka = 0$)

$\Rightarrow$ $C = 0$ and $K = 0$

$$\Rightarrow f(x)=2\left[\frac{x^4}{4}-x^3+x^2\right]=\frac{x^4}{2}-2x^3+2x^2 \Rightarrow f(2) = 8 - 16 + 8 = 0$$

**Hence, the correct option is (2).**

**3.** The mean of the data set comprising of 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data, is:

(1) 16.0

(2) 15.8

(3) 14.0

(4) 16.8

**Solution**

We have,
$n = 16$, $\vec{x} = 16$
$\Rightarrow$ $\Sigma xi = n\overline{x} = 256$

Therefore, new mean, $\overline{x}=\frac{256-16+3+4+5}{18}$ = 14

**Hence, the correct option is (3).**

**4.** The sum of first 9 terms of the series

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+.... \text{ is:}$$

(1) 96

(2) 142

(3) 192

(4) 71

**Solution**

The given series:
$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+....$$

Therefore, $t_n = \dfrac{1^3+2^3+3^3+\cdots+n^3}{1+3+5+\cdots(2n-1)} = \dfrac{\left[\dfrac{n(n+1)}{2}\right]^2}{n^2}$

Therefore, $S_9 = \sum_{n=1}^{9}\dfrac{(n+1)^2}{4} = \dfrac{1}{4}\sum_{k=2}^{10}k^2 \; = \dfrac{1}{4}\left[\dfrac{(10)(11)(21)}{6} - 1\right] = 96$

**Hence, the correct option is (1).**

**5.** Let $O$ be the vertex and $Q$ be any point on the parabola, $x^2 = 8y$. If the point $P$ divides the line segment $OQ$ internally in the ratio 1:3, then the locus of $P$ is:

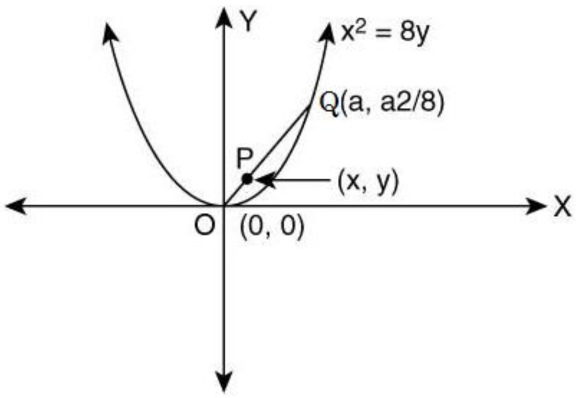
(1) $y^2 = x$

(2) $y^2 = 2x$

(3) $x^2 = 2y$

(4) $x^2 = y$

**Solution**




Since $P$ divides $OQ$ internally in the ratio 1:3.

$\Rightarrow \; x = \dfrac{a}{4}, \;\; y = \dfrac{\dfrac{a^2}{8}}{4} = \dfrac{a^2}{2^2}$

$\Rightarrow \; y = \dfrac{1}{2}x^2$

**Hence, the correct option is (3).**

**6.** Let $\alpha$ and $\beta$ be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = a^n - \beta^n$, for $n \geq 1$, then the value of $\dfrac{a_{10} - 2a_8}{2a_9}$ is equal to:

(1) −6

(2) 3

(3) −3

(4) 6

**Solution**

$\alpha, \beta$ are roots of $x^2 - 6x - 2 = 0$

$\Rightarrow \; \alpha + \beta = 6, \; \alpha\beta = -2$

Now, $\alpha^n - \beta^n = a_n$

$$\Rightarrow \frac{a_{10}-2a_8}{2a_9} = \frac{(\alpha^{10}-\beta^{10})-2(\alpha^8-\beta^8)}{2(\alpha^9-\beta^9)} = \frac{\alpha^8(\alpha^2-2)-\beta^8(\beta^2-2)}{2(\alpha^9-\beta^9)}$$

$$= \frac{\alpha^8(6\alpha)-\beta^8(6\beta)}{2(\alpha^9-\beta^9)} \ [\because \ \alpha^2-6\alpha-2=0 \text{ and } \beta^2-6\beta-2=0]$$

$$= \frac{6(\alpha^9-\beta^9)}{2(\alpha^9-\beta^9)} = 3$$

**Hence, the correct option is (2).**

**7.** If 12 identical balls are to be placed in 3 identical boxes, then the probability that one of the boxes contains exactly 3 balls is:

(1) $55\left(\frac{2}{3}\right)^{10}$

(2) $220\left(\frac{1}{3}\right)^{12}$

(3) $22\left(\frac{1}{3}\right)^{11}$

(4) $\frac{55}{3}\left(\frac{2}{3}\right)^{11}$

**Solution**

Choose 3 balls out of 12 in ${}^{12}C_3$ ways and distribute the remaining 9 balls in two boxes in $2^9$ ways.
However total number of possible ways = $(3)^{12}$

Therefore required probability $= \frac{{}^{12}C_3(2)^9}{(3)^{12}} = \frac{12\times 11\times 10}{6}\times\frac{2^9}{3^{12}} = \frac{55}{3}\left(\frac{2}{3}\right)^{11}$

**Hence, the correct option is (4).**
**Note:** This answer is correct only when the boxes are different and not identical.

**8.** A complex number $z$ is said to be unimodular if $|z| = 1$. Suppose $z_1$ and $z_2$ are complex numbers such that $\frac{z_1-2z_2}{2-z_1z_2}$ is unimodular and $z_2$ is not unimodular. Then the point $z_1$ lies on a:

(1) straight line parallel to $y$-axis.

(2) circle of radius 2.

(3) circle of radius $\sqrt{2}$.

(4) straight line parallel to $x$-axis.

**Solution**

$$\left|\frac{z_1-2z_2}{2-z_1\bar{z}_2}\right| = 1; \ [z_2] \neq 1$$

$$|z_1-2z_2|^2 = |2-z_1\bar{z}_2|^2 \Rightarrow (z_1-2z_2)(\bar{z}_1-\bar{z}_2) = (2-z_1\bar{z}_2)(2-\bar{z}_1z_2)$$

$$\Rightarrow |z_1|^2 - 2z_1\bar{z}_2 - 2z_2\bar{z}_1 + 4|z_2|^2 = 4 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 + |z_1|^2|z_2|^2$$

$$\Rightarrow |z_1|^2 + 4|z_2|^2 = 4|z_1|^2|z_2|^2 \Rightarrow |z_1|^2(1-|z_2|^2) - 4(1-|z_2|^2) = 0$$

$$\Rightarrow (|z_1|^2-4)(1-|z_2|^2) = 0 \Rightarrow |z_1|^2 = 4 \text{ or } |z_2|^2 = 1$$

But $|z_2| \neq 1 \Rightarrow |z_1|^2 = 4$

$\Rightarrow |z_1| = 2$

$\Rightarrow z_1$ lies on a circle of radius 2.

**Hence, the correct option is (2).**

**9.** The integral $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ equals:

(1) $(x^4+1)^{\frac{1}{4}} + c$

(2) $-(x^4+1)^{\frac{1}{4}} + c$

(3) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

(4) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

**Solution**

$$I = \int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$$

Put $x^{-4} = t$

$$\Rightarrow \frac{-4}{x^5}dx = dt \Rightarrow I = \int \frac{-dt}{4(1+t)^{3/4}} = \frac{1}{4}\frac{(1+t)^{1/4}}{1/4} + C = -(1+x^{-4})1/4 + C$$

**Hence, the correct option is (3).**

**10.** The number of points, having both co-ordinates as integers that lie in the interior of the triangle with vertices (0, 0) (0, 41) and (41, 0), is:

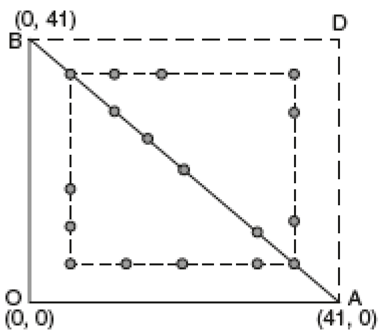
(1) 861

(2) 820

(3) 780

(4) 901

**Solution**




Number of points of desired type = 1 + 2 + 3 + … + 39 $= \frac{40 \times 39}{2} = 780$

**Hence, the correct option is (3).**

**11.** The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x - y + z = 16$, is:

(1) 8

(2) $3\sqrt{21}$

(3) 13

(4) $2\sqrt{14}$

**Solution**

Let $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=\lambda$

$\Rightarrow x = 3\lambda + 2, y = 4\lambda - 1, z = 12\lambda + 2$

If above is the point of intersection, then $(3\lambda + 2) - (4\lambda - 1) + 12\lambda + 2 = 16$

Therefore, $\lambda = 1 \Rightarrow$ point of intersection is $P \equiv (5, 3, 14)$ and $Q \equiv (1, 0, 2)$

$\Rightarrow PQ = \sqrt{16+9+144} = 13$

**Hence, the correct option is (3).**

**12.** The equation of the plane containing the line $2x - 5y + z = 3$; $x + y + 4z = 5$, and parallel to the plane, $x + 3y + 6z = 1$, is:

(1) $x + 3y + 6z = -7$

(2) $x + 3y + 6z = 7$

(3) $2x + 6y + 12z = -13$

(4) $2x + 6y + 12z = 13$

**Solution**

Since plane is parallel to $x + 3y + 6z = 1$

$\Rightarrow$ direction ratios of normal to required plane are <1, 3, 6>

Also plane contains the line

$\{2x - 5y + z = 3; x + y + 4z = 5\}$ (1)

Therefore, the required plane contains the line having direction ratios given by $\frac{x}{-21}=\frac{y}{-7}=\frac{z}{7}$

or <3, 1, −1>

Also point (4, 1, 0) lies on line (1).

Therefore, Equation of the required plane will be

$1(x - 4) + (y - 1)(3) + (z - 0)(6) = 0$ or $x + 3y + 6z = 7$

**Hence, the correct option is (2).**

**13.** The area (in sq. units) of the region described by $\{(x, y) : y^2 \leq 2x \text{ and } y \geq 4x - 1\}$ is:

(1) $\frac{5}{64}$

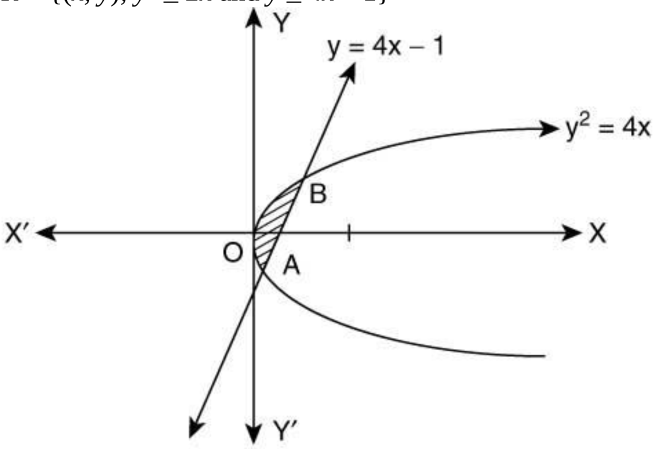
(2) $\frac{15}{64}$

(3) $\frac{9}{32}$

(4) $\frac{7}{32}$

**Solution**

$R = \{(x, y); y^2 \leq 2x \text{ and } y \geq 4x - 1\}$




$R$ = shaded area $\int_{y_A}^{y_B} (x_{\text{line}})dy - \int_{y_A}^{y_B} (x_{\text{parabola}})dy$

$$= \int_{-1/2}^{1} \frac{1}{5}(y+1)dy - \int_{-1/2}^{1} \frac{y^2}{2}dy = \frac{1}{4}\left[\frac{y^2}{2}+y\right]_{-1/2}^{1} - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-1/2}^{1} = \frac{9}{32} \text{ sq.units}$$

**Hence, the correct option is (3).**



**14.** If $m$ is the A.M. of two distinct real numbers $l$ and $n$ ($l$, $n > 1$) and $G_1$, $G_2$ and $G_3$ are three geometric means between $l$ and $n$, then $G_1^4 + 2G_2^4 + G_3^4$ equals.

(1) 1 $lm^2n$

(2) 4 $lmn^2$

(3) 4 $l^2m^2n^2$

(4) 4 $l^2mn$

**Solution**

$m = \frac{l+n}{2}$; (1, $n > 1$) (1)

and $l$, $G_1$, $G_2$, $G_3$ $n$ are in G.P.

$\Rightarrow G_1 = l\left(\frac{n}{l}\right)^{1/4},\ G_2 = l\left(\frac{n}{l}\right)^{1/2},\ G_3 = l\left(\frac{n}{l}\right)^{3/4}$

Therefore, $(G_1)^4 + 2(G_2)^4 + (G_3)^4 = l^4\left(\frac{n}{l} + \frac{2n^2}{l^2} + \frac{n^3}{l^3}\right) = nl(n+1)^2 = 4m^2nl$ (from (1))

**Hence, the correct option is (1).**

**15.** Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$. $k \in R$, is a :

(1) straight line parallel to $y$-axis.

(2) circle of radius $\sqrt{2}$ .

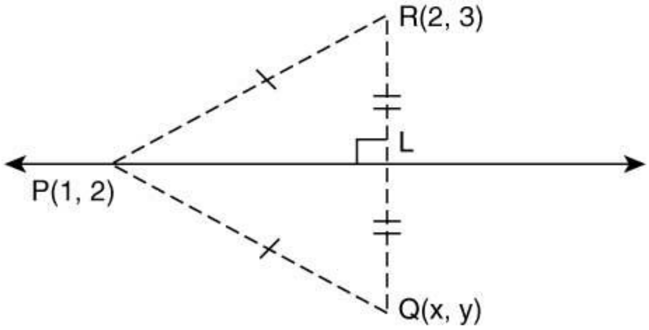
(3) circle of radius $\sqrt{3}$.

(4) straight line parallel to $x$-axis.

**Solution**

Note that $P(1, 2)$ is the point of intersection of given pair of lines. Thus we are to find the locus of image of point $R(2, 3)$ on the given line.
Let $Q(x, y)$ be the image of $R$.




Clearly $PR = PQ$ as $\Delta PRL \sim \Delta PQL$

$\Rightarrow (x-1)^2 + (y-2)^2 = (z-1)^2 + (3-2)^2$

$\Rightarrow (x-1)^2 + (y-2)^2 = (\sqrt{2})^2$

which is a circle of radius $\sqrt{2}$.

**Hence, the correct option is (2).**

**16.** The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is :

(1) 18

(2) $\frac{27}{2}$

(3) 27

(4) $\frac{27}{4}$

**Solution**

Equation of given ellipse is $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$\Rightarrow a^2 = 9, b^2 = 5$

Now, $b^2 = a^2(1 - e^2) \Rightarrow 5 = 9(1 - e^2) \Rightarrow e = 2/3$

One of the end points of the latus recta is $P(ae, b^2/a) \equiv (2, 5/3)$

Equation of tangent to the ellipse at $P$ is $\frac{2x}{9} + \frac{5y}{15} = 1$ or $2x + 3y = 9$ or $\frac{x}{9/2} + \frac{y}{3} = 1$

$\therefore$ Area of quadrilateral $= 4\left(\frac{1}{2}\right)\left(\frac{9}{2}\right)(3) = 27$ sq. units

**Hence, the correct option is (3).**

**17**. The number of integers greater than 6,000 that can be formed, using the digits 3, 5, 6, 7 and 8, without repetition, is:

(1) 192

(2) 120

(3) 72

(4) 216

**Solution**

Any no. greater than 6000 but less than 10,000 that can be formed using the digits, 3, 5, 6, 7 and 8, without repetition has its thousand place digit 6, 7 or 8

Therefore, for the first left place, number of choices = 3

For second left place, number of choices = 4

For third left place, number of choices = 3

For place number of choices = 2

Therefore, number of 4 digit numbers greater than 6000 = 72

Now if we use all the 5 integers the number obtained is definitely greater than 6000, number of such numbers = 5! = 120

Therefore, total numbers formed = 72 + 120 = 192

Now if we used all the 5 integers the number obtained is definitely greater than 6000, number of such numbers = 5! = 120

Therefore, total numbers formed = 72 + 120 = 192

**Hence, the correct option is (1).**

**18.** Let $A$ and $B$ be two sets containing four and two elements respectively. Then the number of subjects for the set $A \times B$, each having at least three elements is:

(1) 256

(2) 275

(3) 510

(4) 219

**Solution**

$n(A) = 4$; $n(B) = 2$

We are to find number of sets of the form $\{x_i, y_i): x_i \in A, y_i \in B$ and $i \geq 3\}$

Case (i) when set contains exactly one element: i.e. $\{(x, y)\}$

$x_1$ has 4 choices and $y_1$ has 2

$\Rightarrow$ 8 sets of such type

Case (ii) When set contains exactly two elements: i.e. $\{(x_1, y_1), (x_2, y_2)\}$

No. of such that $= {}^8C_2 = 28$

Case (iii) Set contains no elements: i.e $< y$ or $\phi$

Therefore, subsets of $(A \times B)$ having at least 3 elements = $(2)^8 - [8 + 28 + 1] = 219$

**Hence, the correct option is (4).**

**19.** Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then a value of $y$ is:

(1) $\frac{3x+x^3}{1-3x^2}$

(2) $\frac{3x-x^3}{1+3x^2}$

(3) $\dfrac{3x+x^3}{1+3x^2}$

(4) $\dfrac{3x-x^3}{1-3x^2}$

**Solution**

Since, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ for $xy<1$.

Now $x.\left(\dfrac{2x}{1-x^2}\right)=\dfrac{2x^2}{1-x^2}=-2\left(\dfrac{1-x^2-1}{1-x^2}\right)=-2+\dfrac{2}{1-x^2}$

Further $|x|<\dfrac{1}{\sqrt{3}} \Rightarrow 0\le x^2<\dfrac{1}{3}\Rightarrow -\dfrac{1}{3}<-x^2\le 0 \Rightarrow \dfrac{2}{3}<1-x^2\le 1$

$\Rightarrow 1\le\dfrac{1}{1-x^2}<\dfrac{3}{2} \Rightarrow 2\le\dfrac{2}{1-x^2}<3\Rightarrow 0\le -2+\dfrac{2}{1-x^2}<1$

$\Rightarrow x.\left(\dfrac{2x}{1-x^2}\right)\in(0,1)$

$\therefore \tan^{-1}(x)+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)=\tan^{-1}\left(\dfrac{1+\dfrac{2x}{1-x^2}}{1-\dfrac{2x^2}{1-x^2}}\right)=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)=\tan^{-1}(y)$ (given)

$\Rightarrow y=\left(\dfrac{3x-x^3}{1-3x^2}\right)$

**Hence, the correct option is (4).**

**20.** The integral $\displaystyle\int_2^4\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}\mathrm{d}x$ is equal to:

(1) 4
(2) 1
(3) 6
(4) 2

**Solution**

$$I=\int_2^4\frac{\log x^2 \qquad dx}{\log x^2+\log(36-12x+x^2)}$$

$$=\int_2^4\frac{\log x^2}{\log x^2+\log(6-x)^2}dx$$

$$=\int_2^4\frac{\log x}{\log x+\log(6-x)}dx \qquad (1)$$

$$\Rightarrow I=\int_2^4\frac{\log(2+4-x)}{\log(2+4-x)+\log(x)}dx$$

By using property $\displaystyle\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$\Rightarrow \ I = \int_2^4 \frac{\log(6-x)}{\log(6-x)+\log x}$ (2)

(1) + (2) gives,

$2I = \int_2^4 1dx = 2 \ \Rightarrow \ I = 1$

**Hence, the correct option is (2).**

**21.** The negation of $\sim s \vee (\sim r \wedge s)$ is equivalent to:

(1) $s \wedge (r \wedge \sim s)$

(2) $s \vee (r \vee \sim s)$

(3) $s \wedge r$

(4) $s \wedge \sim r$

**Solution**

Negation of $\sim s \vee (\sim r \wedge s)$ is given by,

$\sim[\sim s \vee (\sim r \wedge s)] = s \wedge \sim(\sim r \wedge s)$

$= s \wedge (r \vee \sim s)$

$= (s \wedge r) \vee (s \wedge \sim s)$

$= (s \wedge r) \vee (C)$; $C$ = contradiction

$= (s \wedge r) \quad [\because \ p \vee c = p]$

**Hence, the correct option is (3).**

**22.** If the angles of elevation of the top of a tower from three collinear points *A*, *B* and *C*, on a line leading to the foot of the tower, are 30°, 45° and 60° respectively, then the ratio, *AB* : *BC*, is:

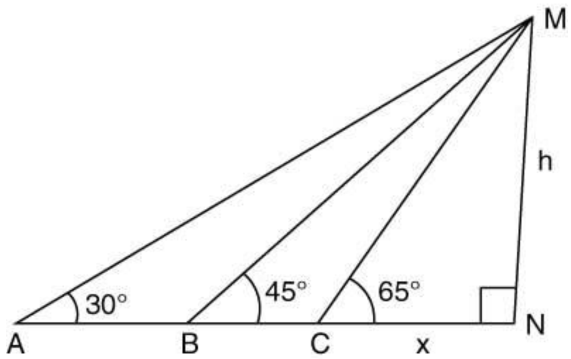
(1) $\sqrt{3} : \sqrt{2}$

(2) $1 : \sqrt{3}$

(3) 2 : 3

(4) $\sqrt{3} : 1$

**Solution**




$\Rightarrow \ \tan 60° = \frac{h}{x} \ \Rightarrow \ x = h \cot 60°$ (1)

$\tan 45° = \frac{h}{BC + x} \ \Rightarrow \ BC = h - h \cot 60°$ (2)

and $\tan 30° = \dfrac{h}{AB+h} \Rightarrow AB = h\cot 30° - h$ (3)

$$\therefore\ AB:BC = \frac{\cot 30° - 1}{1-\cot 60°} = \frac{\sqrt{3}-1}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{1}$$

**Hence, the correct option is (4).**

**23.** $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to:

(1) 3

(2) 2

(3) $\frac{1}{2}$

(4) 4

**Solution**

$$\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x} = \lim_{x\to 0}\frac{(2\sin^2 x)(3+\cos x)}{4x^2\left(\frac{\tan 4x}{4x}\right)} = \frac{1}{2}\lim_{x\to 0}(3+\cos x) = 2$$

**Hence, the correct option is (2).**

**24.** Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $(\vec{a}\times\vec{b})\times\vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$. If $\theta$ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of sin $\theta$ is:

(1) $\frac{-\sqrt{2}}{3}$

(2) $\frac{2}{3}$

(3) $\frac{-2\sqrt{3}}{3}$

(4) $\frac{2\sqrt{2}}{3}$

**Solution**

$$(\vec{a}\times\vec{b})\times\vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a} \Rightarrow -\vec{c}\times(\vec{a}\times\vec{b}) = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a} \Rightarrow -\vec{c}(\vec{b}\cdot\vec{a}) + \vec{c}(\vec{a}\cdot\vec{b}) = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$$

$$\Rightarrow \left[\frac{1}{3}|\vec{b}||\vec{c}| + (\vec{c}\cdot\vec{b})\right]\vec{a} = (\vec{c}\cdot\vec{a})\vec{b}$$

Since $\vec{a}$ and $\vec{b}$ are not collinear,

$$\frac{1}{3}|\vec{b}||\vec{c}| + (\vec{c}\cdot\vec{b}) = 0 \text{ and } \vec{c}\cdot\vec{a} = 0$$

$$\Rightarrow\ \cos\theta + \frac{1}{3} = 0 \Rightarrow \cos\theta = -\frac{1}{3} \Rightarrow \sin\theta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

**Hence, the correct option is (4).**

**25.** If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^{T} = 9I$, where $I$ is $3 \times 3$ identity matrix, then the ordered pair $(a, b)$ is equal to:

(1) $(-1, 1)$

(2) $(2, 1)$

(3) $(-2, -1)$

(4) $(2, -1)$

**Solution**

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}; A \cdot A^{T} = 9I$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & (a+4+2b) \\ 0 & 9 & (2a+2-2b) \\ (a+4+2b) & (2a+2-2b) & (a^2+4+b^2) \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow a + b = -4; a^2 + b^2 + 4 = 9; 2a - 2b = -2;$

$\Rightarrow a = -2, b = -1 \Rightarrow (a, b) \equiv (-2, -1)$

**Hence, the correct option is (3).**

**26.** If the function. $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \le x \le 3 \\ mx+2, & 3 < x \le 5 \end{cases}$ is differentiable, then the value of $k + m$ is :

(1) $\frac{16}{5}$

(2) $\frac{10}{3}$

(3) 4

(4) 2

**Solution**

$$g(x) = \begin{cases} k\sqrt{x+1}; & 0 \le x \le 3 \\ mx+2; & 3 < x \le 5 \end{cases}$$

$g(3^{-}) = 2k; g(3^{+}) = 3m + 2; g(3) = 2k$

$\Rightarrow 2k = 3m + 2$ (1)

Also, $g'(x) = \begin{cases} \dfrac{k}{2\sqrt{x+1}}; & 0 < x < 3 \\ m; & 3 < x < 5 \end{cases}$

$\Rightarrow g'(3^{-}) = \frac{k}{4}; \ g'(3^{+}) = m$

$\Rightarrow \frac{k}{4} = m \Rightarrow k = 4m$

Therefore, from (1), $m = \frac{2}{5},\ \ k = \frac{8}{5}\ \Rightarrow\ k + m = 2$

**Hence, the correct option is (4).**

**27.** The set of all values of $\lambda$ for which the system of linear equations:

$2x_1 - 2x_2 + x_3 = \lambda x_1$

$2x_1 - 3x_2 + 2x_3 = \lambda x_2$

$-x_1 + 2x_2 = \lambda x_3$

has a non-trivial solution,

(1) is a singleton.

(2) contains two elements.

(3) contains more than two elements.

(4) is an empty set.

**Solution**

$2x_1 - 2x_2 + x_3 = \lambda x_1$
$2x_1 - 3x_2 + 2x_3 = \lambda x_2$

$-x_1 + 2x_2 = \lambda x_3$

has a non-trivial solution if $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow\ (2 - \lambda)\ [(3\lambda + \lambda^2) - 4] + 2(-2\lambda + 2) + 1(4 - 3 - \lambda) = 0$

$\Rightarrow\ \lambda \in <1, -3>$

**Hence, the correct option is (2).**

**28.** The normal to the curve, $x^2 + 2xy - 3y^2 = 0$ at (1, 1):

(1) meets the curve again in the second quadrant.

(2) meets the curve again in the third quadrant.

(3) meets the curve again in the fourth quadrant.

(4) does not meet the curve again.

**Solution**

Given curve is, $x^2 + 2xy - 3y^2 = 0$

$\Rightarrow\ 2x + 2x\frac{dy}{dx} + 2y - 6y\frac{dy}{dx} = 0$

$\Rightarrow\ (2x - 6y)\frac{dy}{dx} = -2(x + y)$

$\Rightarrow\ \frac{dy}{dx} = \frac{(x+y)}{(3y-x)} = 1$ at (1, 1)

$\Rightarrow\ -\frac{dx}{dy} = -1$ at (1, 1)

This means equation of normal to given curve is $(y - 1) = -1(x - 1)$

$\Rightarrow x + y = 2$

Substitute $y = 2 - x$ in $x^2 + 2xy - 3y^2 = 0$

$\Rightarrow x^2 + 2x(2 - x) - 3(x - x)^2 = 0$

$\Rightarrow (x - 1)(x - 3) = 0$

$\Rightarrow x = 1$ or $x = 3$

So it will again intersect the given curve at (3, 3), (3, −1), $\left(1, \frac{-1}{3}\right)$ i.e. again meet in the 1st or 4th quadrant.

**Hence, the correct option is (3).**

**29.** The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is:

(1) 2

(2) 3

(3) 4

(4) 1

**Solution**

$C_1 : x^2 + y^2 - 4x - 6y - 12 = 0$
$C_2 : x^2 + y^2 + 6x + 18y + 26 = 0$

$C_1(2, 3);\ r_1 = 5$

$C_2(-3, -9);\ r_2 = 8$

and $C_1C_2 = \sqrt{(2+3)^2 + (3+9)^2} = 13 = r_1 + r_2$

The two circles touch each other externally.

Hence three common tangents can be drawn.

**Hence, the correct option is (2).**

**30.** Let $y(x)$ be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x\log x, (x \ge 1)$. Then $y(e)$ is equal to:

(1) 0

(2) 2

(3) 2e

(4) e

**Solution**

$$(x\log x)\frac{dy}{dx} + y = 2x\log x;\ (x \ge 1)$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x\log x}\right)y = 2$$

It is a linear differential equation of 1st order of the form $\frac{dy}{dx} + Py = Q$

$$\Rightarrow P = \frac{1}{x\log x},\ Q = 2$$

**Note:** At $x = 1$, $P$ is not defined, hence this question is conceptually not correct.

So, $\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x\log x}dx}$

$$= e^{\int \frac{dt}{t}} = e^{\ln t} = t = \log x$$

Therefore, solution of given differential equation is given by

$$y.(\text{I.F}) = \int Q.(\text{I.F})dx + C$$

$$\Rightarrow \quad y(\log x) = \int 2\log x,\, dx + C$$

$$= 2\left[(\log x)x - \int \frac{1}{x}.x\, dx\right] + C$$

$$= 2x \log x - 2x + C$$

when $x = 1$,

$\Rightarrow$ $y(\log 1) = 2 \log 1 - 2 + C$

$\Rightarrow$ $0 = -2 + C \Rightarrow C = 2$

**Note:** Since we need to put $x = 1$ in order to find the value of constant and at $x = 1$, $P$ is not defined, so this whole question is conceptually incorrect. Although if we still solve it we get,

$y(\log x) = 2x \log x - 2x + 2$ as the general solution.

$\Rightarrow$ $y(e) = 2e - 2e + 2$

$\Rightarrow$ $y(e) = 2$

**Hence, the correct option is (2).**