JEE MAIN – 2016

MATHEMATICS

1. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary is

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $\frac{\pi}{3}$

(3) $\frac{\pi}{6}$ (4) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

Solution

We have

$$\text{Re}\left(\frac{2+3i\sin\theta}{1-2i\sin\theta}\right)=0$$

That is,

$$\text{Re}\left(\frac{(2+3i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}\right)=0$$

$$\text{Re}\left(\frac{(2-6\sin 2\theta)+i(5\sin\theta)}{1+4\sin^2\theta}\right)=0$$

$$\frac{2-6\sin^2\theta}{1+4\sin^2\theta}=0$$

$$6\sin^2\theta=2$$

$$\Rightarrow \sin^2\theta=\frac{1}{3}$$

$$\Rightarrow \theta=\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Hence, the correct option is (1).

2. The system of linear equations

$x+\lambda y-z=0$

$\lambda x-y-z=0$

$x+y-\lambda z=0$

has a non-trivial solution for:

(1) exactly three values of λ. (2) infinitely many values of λ.

(3) exactly one value of λ. (4) exactly two values of λ.

Solution

For non-trivial solution, we have

$$\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow 1(\lambda + 1) - \lambda(-\lambda^2 + 1) - 1(\lambda + 1) = 0$$

$$\Rightarrow \lambda(\lambda^2 - 1) = 0$$

$$\Rightarrow \lambda = -1, 0, 1$$

Hence, the correct option is (1).

3. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side x units and a circle of radius r units. If the sum of the areas of the square and the circle so formed is minimum, then

(1) $2x = r$ (2) $2x = (\pi + 4)r$

(3) $(4 - \pi)x = \pi r$ (4) $x = 2r$

Solution

Side length of the square = x.

Radius of circle = r.

Perimeter of square + Perimeter of circle = 2

$$4x + 2\pi r = 2$$

$$\Rightarrow 2x + \pi r = 1$$

The sum of the area is

$$x^2 + \pi r^2 = A$$

$$\pi r = 1 - 2x$$

$$\Rightarrow r = \left(\frac{1-2x}{\pi}\right)$$

Now,
$$A = x^2 + \frac{\pi(1-2x)^2}{\pi^2}$$

$$= x^2 + \frac{(1-2x)^2}{\pi}$$

Therefore,

$$\frac{dA}{dx} = 2x + \frac{1}{\pi}2(1-2x)(-2)$$

$$= 2x - \frac{4}{\pi}(1-2x) = 0$$

Hence,

$$\frac{d^2A}{dx^2} = 2x + \frac{4}{\pi}(2) > 0$$

Now,
$$x - \frac{2}{\pi}(1-2x) = 0$$

$$\pi x - 2 + 4x = 0$$

$$\Rightarrow x = \left(\frac{2}{\pi + 4}\right)$$

The minimum value occurs at

$$x = \frac{2}{\pi + 4}$$

That is,

$$\pi r = 1 - \frac{4}{\pi + 4} = \frac{\pi}{\pi + 4}$$

$$\Rightarrow r = \frac{1}{(\pi + 4)}$$

$$\Rightarrow x = 2r$$

Hence, the correct option is (4).

4. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 min from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then, the time taken (in minutes) by him, from B to reach the pillar, is

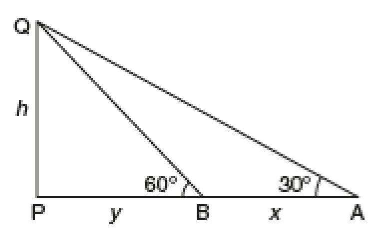
(1) 5 (2) 6

(3) 10 (4) 20

Solution

The given situation is depicted in the following figure. We have

AB = x; BP = y; PQ = h



$$\tan 30° = \frac{h}{x+y} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{x+y} \Rightarrow x + y = h\sqrt{3}$$

$$\tan 60° = \frac{h}{y} \Rightarrow h = y\sqrt{3}$$

Now, $x + y = \sqrt{3}y \times \sqrt{3} = 3y$

$$\Rightarrow x = 2y$$

Let the speed of man be u. Therefore,

$$\left.\begin{array}{l} u = \dfrac{x}{10} \Rightarrow 10 = \dfrac{x}{4} \\ u = \dfrac{y}{t} \Rightarrow t = \dfrac{y}{4} \end{array}\right] \Rightarrow \frac{10}{t} = \frac{x}{y} = \frac{2y}{y} \Rightarrow t = 5 \text{ min}$$

Hence, the correct option is (1).

5. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is NOT true?

(1) E_1, E_2 and E_3 are independent.   (2) E_1 and E_2 are independent.

(3) E_2 and E_3 are independent.   (4) E_1 and E_3 are independent.

Solution

We have

$E_1 \rightarrow$ Dice A shows 4

$E_2 \rightarrow$ Dice B shows 2

$E_3 \rightarrow$ Sum odd (No. on A to be odd + No. on B to be even or No. on B to be odd + No. on A to be even)

Therefore,

$$P(E_1) = \frac{1}{6}$$

$$P(E_2) = \frac{1}{6}$$

$$P(E_3) = \frac{{}^2C_1.{}^3C_1.{}^3C_1}{6.6} = \frac{1}{2}$$

Now.

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

$$P(E_1 \cap E_3) = P(E_1) \cdot P(E_3)$$

$$P(E_2 \cap E_3) = P(E_2) \cdot P(E_3)$$

Thus, the events are independent.

Hence, the correct option is (1).

6. If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true?

(1) $3a^2 - 23a + 44 = 0$ (2) $3a^2 - 26a + 55 = 0$

(3) $3a^2 - 32a + 84 = 0$ (4) $3a^2 - 34a + 91 = 0$

Solution

We have

$$\text{Standard deviation} = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$3.5 = \sqrt{\frac{4+9+a^2+121}{4} - \frac{(16+a)^2}{4^2}}$$

$$\frac{49}{4} = \frac{4(134+a^2)-(16+a)^2}{4^2}$$

$$\Rightarrow 3a^2 - 32a + 84 = 0$$

Hence, the correct option is (3).

7. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then

(1) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$.

(2) g is not differentiable at $x = 0$.

(3) $g'(0) = \cos(\log 2)$.

(4) $g'(0) = -\cos(\log 2)$.

Solution

We have

$$f(x) = |\log 2 - \sin x|$$
$$g(x) = f(f(x))$$

Therefore,

$$g(x) = |\log 2 - \sin f(x)|$$
$$f(x) = \log 2 - \sin x \qquad (\log 2 - \sin x > 0 \text{ in neighbourhood of } x = 0)$$

That is,

$$g(x) = |\log 2 - \sin(\log 2 - \sin x)$$
$$= \log 2 - \sin(\log 2 - \sin x) \qquad [g(x) \text{ is contant function at } x = 0]$$

Therefore,

$$g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$$
$$= \cos x \cdot \cos(\log 2 - \sin x)$$

Substituting $x = 0$, we get

$$g'(0) = \cos(\log 2)$$

Hence, the correct option is (3).

8. The distance of the point (1, −5, 9) from the plane $x - y + z = 5$ measured along the line $x = y = z$ is

(1) $\frac{20}{3}$ (2) $3\sqrt{10}$

(3) $10\sqrt{3}$ (4) $\frac{10}{\sqrt{3}}$

Solution

The line passing through the point (1, 5, 9) and parallel to the line $x = y = z$ is given by

$$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1}$$

Any point on this line is

$$x - 1 = y + 5 = z - 9 = \lambda$$

Now, $(1 + \lambda, \lambda - 5, 9 + \lambda)$ lies on the given plane. Therefore,

$$x - y + z = 5$$
$$\lambda + 1 - \lambda + 5 + \lambda + 9 = 5$$
$$\lambda + 15 = 5$$
$$\Rightarrow \lambda = -10$$

The point is (−9, −15, −1). The distance between the points is

$$\sqrt{(1+9)^2+(-5+15)^2+(9+1)^2} = 10\sqrt{3}$$

Hence, the correct option is (3).

9. The eccentricity of the hyperbola, whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is

(1) $\sqrt{3}$ (2) $\frac{4}{3}$

(3) $\frac{4}{\sqrt{3}}$ (4) $\frac{2}{\sqrt{3}}$

Solution

Length of the latus rectum of hyperbola is 8.

$$\frac{2b^2}{a} = 8 \Rightarrow b^2 = 4a$$

$$\text{Length of conjugate axis} = \frac{1}{2} \times \text{(Distance between the foci)}$$

That is,

$$2b = ae$$

$$b^2 = a^2(e^2 - 1)$$

$$\frac{b^2}{a^2} = e^2 - 1 \Rightarrow \frac{e^2}{4} = e^2 - 1 \Rightarrow 1 = \frac{3e^2}{4} \Rightarrow e = \frac{2}{\sqrt{3}}$$

Hence, the correct option is (4).

10. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then, the equation of the circle, passing through C and having its centre at P is

(1) $x^2 + y^2 - 4x + 9y + 18 = 0$

(2) $x^2 + y^2 - 4x + 8y + 12 = 0$

(3) $x^2 + y^2 - x + 4y - 12 = 0$

(4) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

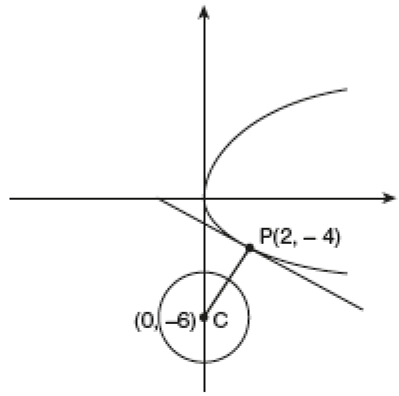
Solution

The parabola is

$$y^2 = 8x$$

Therefore,

$$4a = 8 \Rightarrow a = 2$$



The normal of the parabola that meets at points $P(am_1^2 - 2am)$ and $P(2m_1^2 - 4m)$ given by

$$y = mx - 4m - 2m^3$$

passes through the centre of the circle (0, −6).

$$-6 = -4m - 2m^3$$

$$2m^3 + 4m - 6 = 0$$

$$m^3 + 2m - 3 = 0$$

Therefore,

$$m^2(m-1) + m(m-1) + 3(m-1) = 0$$

$$\Rightarrow m = 1 \text{ and } m^2 + m + 3 \neq 0$$

Hence, the point on the parabola is P(2, –4). Therefore,

$$\text{CP} = \text{Radius} = \sqrt{(0-2)^2 + (-6+4)^2} = \sqrt{4+4} = 2\sqrt{2}$$

The equation of circle is

$$(x-2)^2 + (y+4)^2 = 8$$

$$\Rightarrow x^2 + y^2 - 4x + 8y + 12 = 0$$

Hence, the correct option is (2).

11. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and AadjA = AA^T, then $5a + b$ is equal to

(1) 13 (2) −1

(3) 5 (4) 4

Solution

We have the matrix

$$A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$$

$$A\text{adj}A = AA^T \qquad (1)$$

That is,

$$\text{adj } A = \begin{bmatrix} 2 & b \\ -3 & 5a \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$$

Substituting these values in Eq. (1), we get

$$\begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & b \\ -3 & 5a \end{bmatrix} = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$$

$$\begin{bmatrix} 10a+3b & 5ab-5ab \\ 0 & 3b+10a \end{bmatrix} = \begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix}$$

$$\begin{bmatrix} 10a+3b & 0 \\ 0 & 10a+3b \end{bmatrix} = \begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix}$$

That is, $10a + 3b = 25a^2 + b^2$ (2)

$10a + 3b = 13$ (3)

and $15a - 2b = 0$ (4)

$\Rightarrow b = \dfrac{15a}{2}$ (5)

Substituting the value of b in Eq. (3), we get

$$10a + 3\left(\frac{15a}{2}\right) = 13$$

$$\Rightarrow \frac{20a + 45a}{2} = 13$$

$$\Rightarrow 65a = 13 \times 2$$

$$\Rightarrow a = \frac{2}{5}$$

Substituting the value of a in Eq. (5), we get

$$b = \frac{15}{2} \times \frac{2}{5} = 3$$

$$5a + b = 5\left(\frac{2}{5}\right) + 3 = 5$$

Hence, the correct option is (3).

12. Consider $f(x) = \tan^{-1}\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \dfrac{\pi}{2}\right)$. A normal to $y = f(x)$ at $x = \dfrac{\pi}{6}$ also passes through the point

(1) $\left(\dfrac{\pi}{4}, 0\right)$ (2) (0, 0)

(3) $\left(0, \dfrac{2\pi}{3}\right)$ (4) $\left(\dfrac{\pi}{6}, 0\right)$

Solution

We have

$$f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), \quad x \in \left(0, \frac{\pi}{2}\right)$$

$$f(x) = \tan^{-1}\left(\sqrt{\frac{[\cos(x/2)+\sin(x/2)]^2}{[\cos(x/2)-\sin(x/2)]^2}}\right)$$

$$f(x) = \tan^{-1}\left(\frac{|\cos(x/2)+\sin(x/2)|}{|\cos(x/2)-\sin(x/2)|}\right), \quad x \in \left(0, \frac{\pi}{2}\right)$$

$$f(x) = \tan^{-1}\left[\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}\right]$$

$$f(x) = \tan^{-1}\left[\frac{1+\tan(x/2)}{1-\tan(x/2)}\right]$$

$$f(x) = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$$

$$f(x) = \frac{\pi}{4}+\frac{x}{2}$$

$$f\left(\frac{\pi}{6}\right) = \frac{\pi}{4}+\frac{\pi}{12} = \frac{3\pi+\pi}{12} = \frac{\pi}{3}$$

The point is $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$. Therefore,

$$f'(x) = \frac{1}{2}$$

$$\Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

The slope of normal is −2 and the equation of normal is

$$y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right)$$

$$= -2x + \frac{\pi}{3}$$

Therefore,

$$y = -2x + \frac{2\pi}{3}$$

$$2x + y = \frac{2\pi}{3}$$

$$\Rightarrow \frac{x}{\pi/3} + \frac{y}{2\pi/3} = 1$$

passes through the point $\left(0, \frac{2\pi}{3}\right)$.

Hence, the correct option is (3).

13. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at (−1, −2), then which one of the following is a vertex of this rhombus?

(1) $\left(-\frac{10}{3}, \frac{7}{3}\right)$ (2) $(-3, -9)$

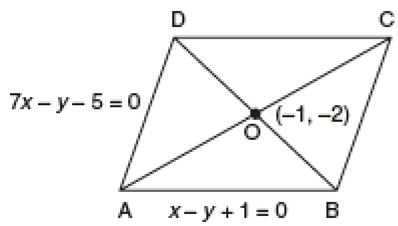
(3) $(-3, -8)$ (4) $\left(\frac{1}{3}, -\frac{8}{3}\right)$

Solution

We have

$$\begin{array}{l} 7x - y - 5 = 0 \\ x - y + 1 = 0 \\ - \quad + \quad - \\ \hline 6x - 6 = 0 \end{array}$$

$$\Rightarrow x = 1$$



$$x - y + 1 = 0$$
$$1 - y + 1 = 0 \Rightarrow y = 2$$

Therefore, point (1, 2), that is, point A(1, 2).

C(α, β); O is mid-point of points A and C.

$$\frac{\alpha + 1}{2} = -1 \Rightarrow \alpha + 1 = -2 \Rightarrow \alpha = -3$$

$$\frac{\beta + 2}{2} = -2 \Rightarrow \beta + 2 = -4 \Rightarrow \beta = -6$$

Therefore, we get point C(−3, −6).

Point B is (x_1, y_1) and point D is (x_2, y_2). That is,

$$x_1 - y_1 + 1 = 0 \text{ and } 7x_2 - y_2 - 5 = 0$$

$$\frac{x_1 + x_2}{2} = -1 \text{ and } \frac{y_1 + y_2}{2} = -2$$

$$x_2 = -2 - x_1 \text{ and } y_2 = -4 - 2y_1$$

Therefore,

$$7(-2 - x_1) + 4 + y_1 - 5 = 0$$
$$-14 - 7x_1 + y_1 - 1 = 0$$
$$-7x_1 + y_1 - 15 = 0$$
$$-7x_1 + x_1 + 1 - 15 = 0$$
$$-6x_1 - 14 = 0$$

$$\Rightarrow x_1 = -\frac{7}{3} \text{ and } y_1 = \frac{-7}{3} + 1 = \frac{-4}{3}$$

Therefore, we get point $\mathrm{B}\left(-\frac{7}{3},-\frac{4}{3}\right)$.

Now,

$$x_2 = -2+\frac{7}{3} = +\frac{1}{3}$$

$$y_2 = -4+\frac{4}{3} = -\frac{8}{3}$$

Therefore, we get point $\mathrm{D}\left(\frac{1}{3},\frac{-8}{3}\right)$.

Hence, the correct option is (4).

14. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)dx = xdy$, then $f\left(-\frac{1}{2}\right)$ is equal to

(1) $\frac{4}{5}$

(2) $-\frac{2}{5}$

(3) $-\frac{4}{5}$

(4) $\frac{2}{5}$

Solution

We have

$$y(1 + xy)dx = xdy$$

$$\Rightarrow ydx - xy^2dx = xy^2dx$$

$$\Rightarrow ydy - ydx = xy^2dx$$

$$\Rightarrow ydx - xdy = -xy^2dx$$

$$\Rightarrow \left(\frac{y\,dx - x\,dy}{y^2}\right) = -x\,dx$$

Therefore,

$$\int d\left(\frac{x}{y}\right) = -\int x\,dx$$

$$\Rightarrow \frac{x}{y} = -\frac{x^2}{2} + C \quad (1)$$

which passes through $(1, -1)$.

$$-1 = -\frac{1}{2} + C$$

$$C = -1 + \frac{1}{2} = -\frac{1}{2}$$

Substituting the value of C in Eq. (1), we get

$$\frac{x}{y} = -\frac{x^2}{2} - \frac{1}{2}$$

$$\frac{x}{y} = -\frac{(x^2+1)}{2}$$

$$\Rightarrow y = \frac{-2x}{x^2+1} \text{ and } x = -\frac{1}{2}$$

Therefore,

$$f\left(-\frac{1}{2}\right) = -\frac{-2(-1/2)}{1+(1/4)} = \frac{4}{5}$$

Hence, the correct option is (1).

15. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary; then the position of the word SMALL is

(1) 58th (2) 46th

(3) 59th (4) 52nd

Solution

For the word SMALL:

- The number of word starting from A is

$$\frac{4!}{2!} = 12$$

- The number of words starting from L is

$$4! = 24$$

- The number of words starting from M is

$$\frac{4!}{2!} = 12$$

- The number of words starting with SA is

$$\frac{3!}{2!} = 3$$

- The number of words starting with SL is

$$3! = 6$$

- The number of words starting with SMALL is 1.

Therefore, the position of the word SMALL to occur is 12 + 24 + 12 + 3 + 6 + 1 = 58th position.

Hence, the correct option is (1).

16. If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is

(1) $\frac{7}{4}$ (2) $\frac{8}{5}$

(3) $\frac{4}{3}$ (4) 1

Solution

Let a and b be the 1st term and the common difference of A.P., namely, $t_2 = a + d$, $t_5 = a + 4d$ and $t_9 = a + 8d$ are in G.P.

That is, t_2, t_5 and t_9 in G.P.:

$$\frac{t_5}{t_2} = \frac{t_9}{t_5} \Rightarrow t_5^2 = t_2 t_9$$

$$\Rightarrow (a + 4d)^2 = (a + d)(a + 8d)$$

$$\Rightarrow a^2 + 16d^2 + 8ad = a^2 + 8ad + ad + 8d^2$$

$$\Rightarrow 8d^2 - ad = 0$$

$$\Rightarrow d(8d - a) = 0$$

$$\Rightarrow d = 0$$

which is not possible and $a = 8d$.

Now, the common ratio of G.P. is

$$\frac{t_5}{t_2} = \frac{a+4d}{a+d} = \frac{12d}{9d} = \frac{4}{3}$$

Hence, the correct option is (3).

17. If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all terms in this expansion, is

(1) 729 (2) 64

(3) 2187 (4) 243

Solution

We have

$$\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$$

The number of terms in the expansion is

$${}^{n+2}C_2 = 28$$

$$\Rightarrow \frac{(n+2)(n+1)}{2} = 28$$

$$\Rightarrow (n + 2)(n + 1) = 56$$

$$\Rightarrow n^2 + 3n - 54 = 0$$

$$\Rightarrow n = 6$$

The sum of coefficient is

$$(1 - 2 + 4)^6 = 3^6 = 729$$

This is possible only when we are not considering the number of dissimilar term.

Note: If we consider the dissimilar term, then number of terms is $2n + 1$ and hence

$$2n + 1 = 28 \Rightarrow n = \frac{27}{2}$$

which is not possible (and hence the question may be considered wrong).

Hence, the correct option is (1).

18. If the sum of the first ten terms of the series

$\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \cdots$, is $\frac{16}{5}$m, then m is equal to

(1) 99
(2) 102
(3) 101
(4) 100

Solution

We have

$$\left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + \left(\frac{20}{5}\right)^2 + \cdots \text{ upto 10 terms } = \frac{16}{5}\text{m}$$

That is,

$$\left(\frac{4}{5}\right)^2 (2^2 + 3^2 + 4^2 + 5^2 + \cdots + 11^2) = \frac{16}{5}\text{m}$$

$$\Rightarrow 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + 1^2 = 5m + 1^2$$

$$\Rightarrow \frac{11(11+1)[2(11)+1]}{6} = 5m + 1$$

$$\Rightarrow 22 \times 23 = 5m + 1$$

$$\Rightarrow 506 = 5m + 1$$

$$\Rightarrow 5m = 505$$

$$\Rightarrow m = 101$$

Hence, the correct option is (3).

19. If the line, $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane, $lx + my - z = 9$, then $l^2 + m^2$ is equal to

(1) 2
(2) 26
(3) 18
(4) 5

Solution

The line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane $lx + my - z = 9$.

Point (3, −2, −4) lies in the plane and DRs (direction ratios) of the line of plane are at 90°.

Therefore,

$$l(3) + m(-2) + 4 = 0$$

$$\Rightarrow 3l - 2m = 5$$

$$\Rightarrow l(2) + m(-1) + 3(-1) = 0$$

$$\Rightarrow 2l - m = 3$$

$$\Rightarrow m = 2l - 3$$

Now, $3l - 2(2l - 3) = 5$

$$\Rightarrow 3l - 4l + 6 = 5$$

$$\Rightarrow -l = -1 \Rightarrow l = 1$$

$$\Rightarrow m = 2 - 3 = -1$$

$$\Rightarrow l = 1 \text{ and } m = -1$$

Therefore, $l^2 + m^2 = 2$.

Hence, the correct option is (1).

20. The Boolean Expression $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$ is equivalent to

(1) $p \vee \sim q$ (2) $\sim p \wedge q$

(3) $p \wedge q$ (4) $p \vee q$

Solution

By Boolean expression, we have

$$(p \wedge \sim q) \vee q \vee (-p \wedge q)$$

Therefore,

$$(p \wedge \sim q) \vee q = (p \vee q) \wedge (\sim q \vee q) = (p \vee q) \wedge t = (p \vee q)$$

Now, $(p \vee q) \vee (\sim p \wedge q) = (p \vee q)$

Hence, the correct option is (4).

21. The integral $\int \frac{2x^{11} + 5x^9}{(x^5 + x^3 + 1)^3} dx$ is equal to

(1) $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2} + C$ (2) $\frac{-x^5}{(x^5 + x^3 + 1)^2} + C$

(3) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$ (4) $\frac{x^5}{2(x^5 + x^3 + 1)^2} + C$

Solution

We have

$$\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$$

$$\Rightarrow \int \frac{2x^{12} + 5x^9}{x^{15}\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3} dx$$

$$\Rightarrow \int \frac{\left(\frac{2}{x^3} + \frac{5}{x^6}\right)}{\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3} dx$$

$$\Rightarrow 1 + x^{-2} + x^{-5} = t$$

$$\Rightarrow \left(-\frac{2}{x^3} - \frac{5}{x^6}\right) dx = dt$$

$$\Rightarrow \left(\frac{2}{x^3} + \frac{5}{x^6}\right) dx = -dt$$

$$\Rightarrow \int \frac{1}{t^3} dt = -\frac{1}{2t^2} + 6$$

$$= -\frac{1}{2\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^2} + 6 \equiv -\frac{x^{10}}{2(x^5+x^3+1)} + C$$

Hence, the correct option is (3).

22. If one of the diameters of the circle, given by the equation $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (−3, 2), then the radius of S is

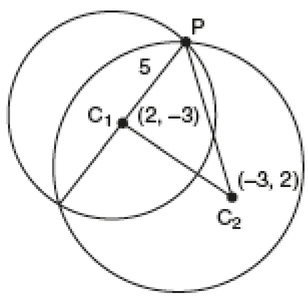
(1) 10 (2) $5\sqrt{2}$

(3) $5\sqrt{3}$ (4) 5

Solution

We have

$$x^2 + y^2 - 4x + 6y - 12 = 0$$
$$\Rightarrow (x^2 - 4x + 4) + (y^2 + 6y + 9) - 25 = 0$$
$$\Rightarrow (x - 2)^2 + (y + 3)^2 = 25$$



Therefore,

$$C_1C_2 = \sqrt{(2+3)^2 + (-3-2)^2} = 5\sqrt{2}$$

$$(C_2P)^2 = (C_1C_2)^2 + (C_1P)^2$$
$$= 50 + 25 = 75$$

Therefore, the required radius of the circle S is

$$C_2P = 5\sqrt{3}$$

Hence, the correct option is (3).

23. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2).....3n}{n^{2n}}\right)^{1/n}$ is equal to:

(1) 3 log3 − 2 (2) $\frac{18}{e^4}$

(3) $\frac{27}{e^2}$ (4) $\frac{9}{e^2}$

Solution

We have

$$y = \lim_{n\to\infty}\left(\frac{(n+1)(n+2)\ldots(n+2n)}{2^{2n}}\right)^{1/n}$$

$$\Rightarrow \ln y = \underset{n\to\infty}{\text{lt}} \frac{1}{n}\left[\ln\left(\frac{n+1}{n}\right)+\ln\left(\frac{n+2}{n}\right)+\ldots+\ln\left(\frac{n+2n}{n}\right)\right]$$

$$= \underset{n\to\infty}{\text{lt}} \frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln\left(1+\frac{2n}{n}\right)\right]$$

$$= \underset{n\to\infty}{\text{lt}} \frac{1}{n}\sum_{r=1}^{2n}\ln\left(1+\frac{r}{n}\right)$$

$$= \int_0^2 \ln(1+x)dx = (x+1)\ln(x+1)-(x+1)\Big|_0^2$$

$$= 3(\ln 3) - 3 + 1$$

$$= 3(\ln 3) - 2 = \ln 27 - \ln e^2$$

$$\ln y = \ln\left(\frac{27}{e^2}\right) \Rightarrow y = \frac{27}{e^2}$$

Hence, the correct option is (3).

24. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on

(1) a parabola
(2) a circle
(3) an ellipse which is not a circle
(4) a hyperbola

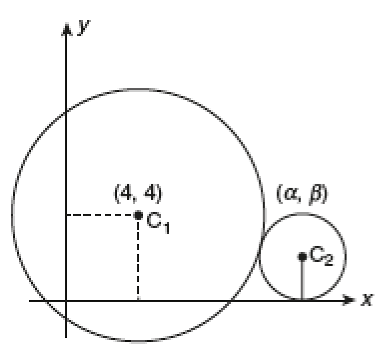
Solution

We have

$$(x^2 - 3x + 16) + (y^2 - 8y + 16) = 36$$

$$\Rightarrow (x-4)^2 + (y-4)^2 = 36$$

The centre of the circle is (α, β) and the radius of the circle is β.



$C_1C_2 = 4 + \beta$

Now,
$$C_1C_2 = 4 + \beta$$

$$\sqrt{(4-\alpha)^2+(4-\beta)^2} = (4+\beta)$$

$$16 + \alpha^2 - 3\alpha + 16 + \beta^2 - 8\beta = \beta^2 + 16 + 8\beta$$

$$(\alpha - 4)^2 = 8\beta - 16$$
$$(\alpha - 4)^2 = 8(\beta - 2)$$
$$(x - 4)^2 = 8(y - 2)$$

which is a parabola.

Hence, the correct option is (1).

25. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{5\pi}{6}$

(2) $\frac{3\pi}{4}$

(3) $\frac{\pi}{2}$

(4) $\frac{2\pi}{3}$

Solution

It is given that $\vec{a}, \vec{b}$ and $\vec{c}$ are three unit vectors such that

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$$

Therefore,

$$(\vec{a} \cdot \vec{c})\bar{b} - (\vec{a} \cdot \vec{b})\vec{c} = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$$

That is,

$$\vec{a} \cdot \vec{c} = \frac{\sqrt{3}}{2} \text{ and } \vec{a} \cdot \vec{b} = -\frac{\sqrt{3}}{2}$$

Here, $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\cos\beta = -\frac{\sqrt{3}}{2}$, where α is angle between $\vec{a}$ and $\vec{c}$; β is the angle between $\vec{a}$ and $\vec{b}$.

Therefore,

$$\beta = \frac{5\pi}{6}$$

Hence, the correct option is (1).

26. Let $p = \lim_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{1/2x}$ then $\log p$ is equal to

(1) $\frac{1}{4}$

(2) 2

(3) 1

(4) $\frac{1}{2}$

Solution

We have

$$p = \lim_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{1/2x}$$

Therefore,

$$p = e^{\lim\limits_{x\to 0^+}(1+\tan^2\sqrt{x}-1)\cdot\frac{1}{2x}}$$

$$= e^{\lim\limits_{x\to 0^+}\frac{\tan^2\sqrt{x}}{2(\sqrt{x})^2}} = e^{1/2}$$

Hence, $\ln p = 1/2$.

Hence, the correct option is (4).

27. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is

(1) 9 (2) 3

(3) 5 (4) 7

Solution

We have

$$(\cos x + \cos 3x) + (\cos 2x + \cos 4x) = 0,\ x \in [0, 2\pi]$$

$$2\cos 2x \cos x + 2\cos 3x \cos x = 0$$

$$2\cos x(\cos 2x + \cos 3x) = 0$$

Therefore,

$$4\cos x \cos\frac{5x}{2}\cos\frac{x}{2} = 0$$

$\cos x = 0$	or	$\cos\frac{5x}{2} = 0$	or	$\cos\frac{x}{2} = 0$
$x = 90°, 270°$		$\frac{5x}{2} = (2n+1)+\frac{\pi}{2}$		$\frac{x}{2} = (2m+1)\frac{\pi}{2}$
		$x = (2n+1)\frac{\pi}{6}$		$x = (2m+1)\pi$
		$x = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$		Required value = π

Therefore, the total number of solutions is 7.

Hence, the correct option is (4).

28. The sum of all real values of x satisfying the equation $(x^2 - 5x + 5)^{x^2+4x-60} = 1$ is

(1) 5 (2) 3

(3) −4 (4) 6

Solution

We have

$$(x^2 - 5x + 5)^{x^2+4x-60} = 1$$

Case 1: $x^2 + 4x - 60 = 0$

$$(x + 10)(x - 6) = 0 \Rightarrow x = -10, 6$$

Case 2: $x^2 - 5x + 5 = 1$ or $x^2 - 5x + 5 = -1$

$x^2 - 5x + 4 = 0$ $\quad$ $x^2 - 5x + 6 = 0$

$x = 1, 4$ $\quad$ $x = 2$ and 3

$x^2 + 4x - 60$ must be even number for these two values.

$4 + 8 - 60 = -48$

$9 + 12 - 60x$

Therefore, let us reject $x = 3$.

Thus, the sum of values is

$$-10 + 6 + 1 + 4 + 2 = 3$$

Hence, the correct option is (2).

29. The area (in sq. units) of the region $\{(x, y): y^2 \geq 2x \text{ and } x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is

(1) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$

(2) $\pi - \frac{4}{3}$

(3) $\pi - \frac{8}{3}$

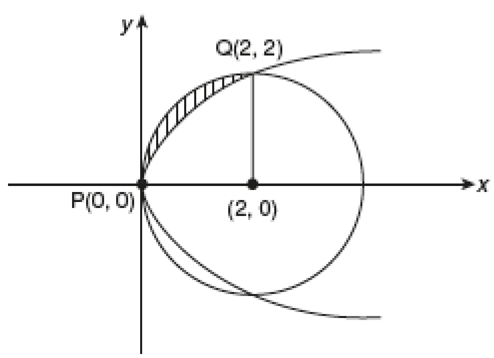
(4) $\pi - \frac{4\sqrt{2}}{3}$

Solution

We have $y^2 - 2x \geq 0$ and $x^2 + y^2 - 4x \leq 0$, $x \geq 0$, $y \geq 0$.

$$(x - 2)^2 + y^2 \leq 4$$

Point of intersection of both curves $y^2 = 2x$ and $(x - 2)^2 + y^2 = 4$ is (0, 0) and (2, 2).



The required area is

$$\int_0^2 (y_1 - y_2)dx = \int_0^2 (\sqrt{4x - x^2} - \sqrt{2x})dx$$

$$= \frac{\pi(2)^2}{4} - \sqrt{2}\int_0^2 \sqrt{x}\, dx$$

$$= \pi - \frac{8}{3}$$

Hence, the correct option is (3).

30. If $f(x)+2f\left(\frac{1}{x}\right)=3x,\ x\neq 0$ and $S=\{x\in\mathbb{R}: f(x)=f(-x)\},$ then S

(1) contains more than two elements.

(2) is an empty set.

(3) contains exactly one element.

(4) contains exactly two elements.

Solution

We have

$$f(x)+2f\left(\frac{1}{x}\right)=3x,\ x\neq 0$$

Replacing x by 1/2, we get

$$f\left(\frac{1}{x}\right)+2f(x)=3/x$$

$$f\left(\frac{1}{x}\right)=\frac{3}{x}-2f(x)$$

$$f(x)+\frac{6}{x}-4f(x)=3x$$

$$\Rightarrow \frac{6}{x}-3x=3f(x)$$

$$\Rightarrow f(x)=\frac{2}{x}-x$$

Now,

$$f(x)=f(-x)$$

$$\frac{2}{x}-x=-\frac{2}{x}+x$$

$$\frac{4}{x}=2x$$

That is,

$$\frac{2}{x}=x\Rightarrow x^2=2\Rightarrow x=-\sqrt{2},\sqrt{2}$$

Therefore, S contains only two elements.

Hence, the correct option is (4).