Solved Mathematics Paper

1. The function $f:\mathbb{R}\rightarrow\left[-\frac{1}{2},\frac{1}{2}\right]$ defined as $f(x)=\frac{x}{1+x^2}$, is:

(1) injective but not surjective. (2) surjective but not injective.

(3) neither injective nor surjective. (4) invertible.

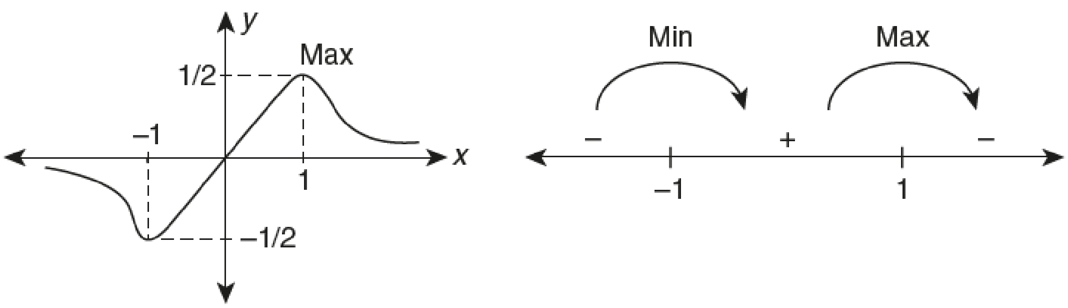
Solution

(2) The given function is defined as

$$f(x)=\frac{x}{1+x^2}$$

Now, $$\frac{dy}{dx}=\frac{(1+x^2)\cdot 1-x(0+2x)}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}=0$$

Therefore, x = 1, –1, which is odd function and there is symmetry about the origin – that is, the function is non-monotonic and non-injective – in the resultant curve as shown in the following figures:



Any line parallel to x-axis cuts the graph more than one point; hence, the function is many-to-one.

Now, $$y=\frac{x}{1+x^2}$$

$$\Rightarrow x^2(y)-x+y=0$$

Now, D > 0; $1-4y^2\geq 0.$ That is, the range is

$$y\in\left[-\frac{1}{2},\frac{1}{2}\right]=\text{codomain}$$

Hence, the function is onto. Therefore, the function is surjective but not injective.

2. If, for a positive integer n, the quadratic equation,

$$x(x+1)+(x+1)(x+2)+\ldots+(x+\overline{n-1})(x+n)=10n$$

has two consecutive integral solutions, then n is equal to:

(1) 9 (2) 10

(3) 11 (4) 12

Solution

(3) The given quadratic equation is

$$x(x+1)+(x+1)(x+2)+\cdots+\left[x+(n-1)\right](x+n)=10n$$

After simplifying, we get

$$nx^2+\{1+3+5+7+\cdots+(2n-1)x+[(0\cdot 1)+(1\cdot 2)+(2\cdot 3)+\cdots+(n-1)n\}=10n$$

$$nx^2+n^2x+\left[\frac{n(n^2-1)}{3}\right]-10n=0$$

$$x^2+nx+\left(\frac{n^2-1-30}{3}\right)=0$$

$$x^2+nx+\left(\frac{n^2-31}{3}\right)=0$$

Using n = 11 (where $n \in \mathrm{I}$), we get

$$x^2+11x+\left(\frac{121-31}{3}\right)=0$$

$$x^2 + 11x + 30 = 0$$

$$(x + 6)(x + 5) = 0$$

Therefore, x = −5, −6 (i.e., two consecutive integral solutions).

Hence, n = 11.

3. Let ω be a complex number such that $2\omega + 1 = z$, where $z=\sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix}=3k$, then k is equal to:

(1) z (2) −1

(3) 1 (4) $-z$

Solution

(4) Applying column-reducing operation, $C_1 = C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 3 & 1 & 1 \\ 0 & -(1+\omega^2) & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix}=3k$$

$$\begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix}=3k$$

Since $1 + \omega + \omega^2 = 0$, open by C_1:

$$3(\omega^2 - \omega^4) = 3k$$

$$3(-1-\omega-\omega) = 3k$$

$$-3(1 + 2\omega) = 3k$$

$$-3z = 3k \Rightarrow k=-z$$

4. If $A=\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj$(3A^2 + 12A)$ is equal to

(1) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (2) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

(3) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ (4) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$

Solution

(1) The given matrix is

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

Therefore, $$A^2 = \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$$

That is, $$3A^2 = \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix}$$

Also, $$12A = \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$$

Hence, $$3A^2 + 12A = \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$$

Therefore, $$\text{adj}(3A^2 + 12A) = \begin{bmatrix} 51 & 68 \\ 84 & 72 \end{bmatrix}$$

5. If S is the set of distinct values of b for which the following system of linear equations

$$x + y + z = 1$$
$$x + ay + z = 1$$
$$ax + by + z = 0$$

has no solution, then S is

(1) an infinite set.

(2) a finite set containing two or more elements.

(3) a singleton.

(4) an empty set.

Solution

(3) For $\Delta = 0$ (and at the one of the solutions of $\Delta_1, \Delta_2, \Delta_3 \neq 0$):

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix} = 0$$

$$1(a - b) - 1(1 - a) + 1(b - a^2) = 0$$
$$2a - b - 1 + b - a^2 = 0$$
$$a^2 - 2a + 1 = 0 \Rightarrow a = 1$$

Using $a = 1$ in the given system of equations, we get

$$x + y + z = 1$$
$$x + y + z = 1$$
$$x + by + z = 0$$

We see that there is only one value of b; therefore, S is singleton set.

6. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then, the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is

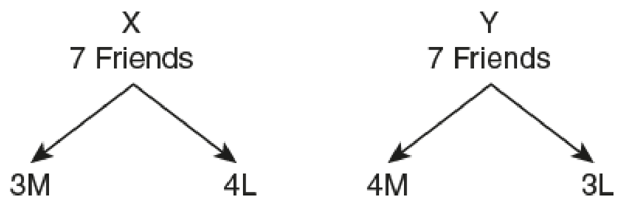
(1) 468 (2) 469

(3) 484 (4) 485

Solution

(4)

The situation is depicted as in the following figure:



- **Case I:** 3L from X side and 3M from Y side. Therefore,
$${}^4C_3 \times {}^4C_3 = 4 \times 4 = 16.$$
- **Case II:** 3M from X side 3L from Y side. Therefore,
$${}^3C_3 \times {}^3C_3 = 1 \times 1 = 1$$
- **Case III:** 2L and 1M from X side and 2M and 1L from Y side. Therefore,
$$({}^4C_2 \times {}^3C_1) \times ({}^4C_2 \times {}^3C_1) = (6 \times 3) \times (6 \times 3) = 18 \times 18 = 324$$
- **Case IV:** 2M and 1L from X side and 1M and 2L from Y side.
$$({}^3C_2 \times {}^4C_1) \times ({}^4C_1 \times {}^3C_2) = (3 \times 4) \times (4 \times 3) = 12 \times 12 = 144$$

Therefore, the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is

Case I + Case II + Case III + Case IV = 16 + 1 + 324 + 144 = 485

7. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \cdots + ({}^{21}C_{10} - {}^{10}C_{10})$ is

(1) $2^{21} - 2^{10}$ (2) $2^{20} - 2^{9}$

(3) $2^{20} - 2^{10}$ (4) $2^{21} - 2^{11}$

Solution

(3)

$$({}^{21}C_1 + {}^{21}C_2 + \cdots + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_{10})$$

$$= \frac{1}{2}[2 \times {}^{21}C_1 + 2 \times {}^{21}C_2 + \cdots + 2 \times {}^{21}C_{10}] - (2^{10} - 1)$$

$$= \frac{1}{2}({}^{21}C_0 + {}^{21}C_1 + {}^{21}C_2 + \cdots + {}^{21}C_{10} + {}^{21}C_{11} + \cdots + {}^{21}C_{20} + {}^{21}C_{21} - ({}^{21}C_0 + {}^{21}C_{21})] - (2^{10} - 1)$$

$$= \frac{1}{2}(2^{21} - 2) - (2^{10} - 1)$$

$$= 2^{20} - 1 - 2^{10} + 1$$

$$= 2^{20} - 2^{10}$$

8. For any three positive real numbers a, b and c,

$$9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c).$$

Then:

(1) b, c and a are in A.P.

(2) a, b and c are in A.P

(3) a, b and c are in G.P.

(4) b, c and a are in G.P

Solution

(1) We have

$$(15a)^2 + (3b)^2 + (5c)^2 - (15a)(5c) - (15a)(3b) - (3b)(5c) = 0$$

$$\frac{1}{2}[(15a-3b)^2 + (3b-5c)^2 + (5c-15a)^2] = 0$$

It is possible when $15a = 3b = 5c$.

Therefore, $b = \frac{5c}{3}$ and $a = \frac{c}{3}$.

That is $a + b = 2c$.

Hence, b, c and a are in A.P.

9. Let $a, b, c \in \mathbb{R}$. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and $f(x+y) = f(x) + f(y) + xy, \ \forall x, y \in \mathbb{R}$, then $\sum_{n=1}^{10} f(n)$ is equal to:

(1) 165 (2) 190

(3) 255 (4) 330

Solution

(4) We have

$$f(x) = x^2 + bx + c$$

$$f(1) = a + b + c = 3$$

Now, $$f(x + y) = f(x) + f(y) + xy$$

Substituting $y = 1$, we get

$$f(x + 1) = f(x) + f(1) + x$$

$$f(x + 1) = f(x) + x + 3$$

Now, $$f(2) = 7 \text{ and } f(3) = 12$$

Therefore,

$$S_n = 3 + 7 + 12 + \cdots + t_n \quad (1)$$

$$S_n = 3 + 7 + \cdots + t_{n-1} + S_n \quad (2)$$

On subtracting Eq. (2) from Eq. (1), we get

$$t_n = 3 + 4 + 5 + \cdots \text{ up to } n \text{ terms.}$$

Therefore, $$t_n = \frac{(n^2 + 5n)}{2}$$

$$S_n = \sum t_n = \sum \frac{(n^2 + 5n)}{2}$$

$$S_n = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{S_n(n+1)}{2}\right]$$

On further simplification, we get $S_n = 330$.

10. $\lim_{x\to\frac{\pi}{2}} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals:

(1) $\frac{1}{16}$ (2) $\frac{1}{8}$

(3) $\frac{1}{4}$ (4) $\frac{1}{24}$

Solution

(1) We have

$$\lim_{x\to\frac{\pi}{2}} \frac{\cot x(1-\sin x)}{-8\left(x-\frac{\pi}{2}\right)^3}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2}-x\right)}{8\left(\frac{\pi}{2}-x\right)} \frac{\left(1-\cos\left(\frac{\pi}{2}-x\right)\right)}{\left(\frac{\pi}{2}-x\right)^2} = \lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2}-x\right)}{8\left(\frac{\pi}{2}-x\right)} \frac{\left(\sin^2\left(\frac{\pi}{2}-x\right)\right)}{2\left(\frac{\pi}{2}-x\right)^2}$$

$$= \frac{1}{8}\times 1\times\frac{1}{2} = \frac{1}{16}$$

11. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}\cdot g(x)$, then $g(x)$ equals:

(1) $\frac{3x\sqrt{x}}{1-9x^3}$ (2) $\frac{3x}{1-9x^3}$

(3) $\frac{3}{1+9x^3}$ (4) $\frac{9}{1+9x^3}$

Solution

(4) We have

$$y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right) = \tan^{-1}\left(\frac{2(3x\sqrt{x})}{1-(3x\sqrt{x})^2}\right)$$

$$= 2\tan^{-1}(3x\sqrt{x})$$

Now, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2}{1+(3x\sqrt{x})^2} 3\left((x)\frac{1}{2\sqrt{x}} + \sqrt{x}(1)\right)$$

$$= \frac{6}{1+9x^3}\left(\frac{\sqrt{x}}{2}+\sqrt{x}\right) = \frac{9\sqrt{x}}{1+9x^3} = \sqrt{x}\left(\frac{9}{1+9x^3}\right) = \sqrt{x}\cdot g(x)$$

Therefore, $g(x) = \frac{9}{1+9x^3}$.

12. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the y-axis passes through the point:

(1) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (2) $\left(\frac{1}{2}, -\frac{1}{3}\right)$

(3) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (4) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

Solution

(1) It is given that

$$y(x-2)(x-3) = x + 6$$

At y-axis, we know that $x = 0$. Therefore,

$$y(-2)(-3) = 0 + 6$$

Now,
$$y(x^2 - 5x + 6) = x + 6$$

$$\Rightarrow y = \frac{x+6}{x^2-5x+6}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{[(x^2-5x+6)(1)]-[(x+6)(2x-5)]}{(x^2-5x+6)^2}$$

At $x = 0$, we have $y = 1$ as follows:

$$y = \frac{6-[(6)(-5)]}{6^2} = 1$$

Therefore, the equation of normal is

$$y - 1 = -1(x - 0)$$

That is, $y + x - 1 = 0$ or $y + x = 1$.

Hence, the normal to the given curve line passes through $\left(\frac{1}{2}, \frac{1}{2}\right)$.

13. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then, the maximum area (in sq.m) of the flower-bed, is:

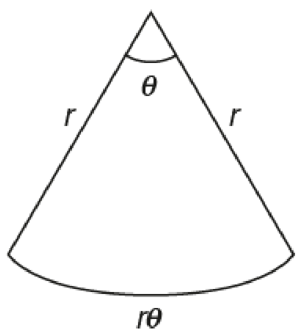
(1) 10 (2) 25

(3) 30 (4) 12.5

Solution

(2) It is given that $r + r + r\theta = 20$ meters. Therefore,

$$\theta = \frac{20-2r}{r}$$



Now, the area is

$$\frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{20-2r}{r}\right)$$

That is, $z = \frac{1}{2}(20r - 2r^2)$

Differentiating w.r.t. r, we get

$$\frac{dz}{dr} = \frac{1}{2}(20 - 4r) = 0$$

$$\Rightarrow r = 5$$

At $r = 5$, we get $\theta = 2$; therefore, $\frac{d^2z}{dr^2} < 0$ (hence, it is maxima).

Therefore, the maximum area is

$$z = \frac{1}{2}r^2\theta = \frac{1}{2} \times 5^2 \times 2 = 25 \text{ m}^2$$

14. Let $I_n = \int \tan^n x\, dx, (n > 1)$. If $I_4 + I_6 = a \tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to:

(1) $\left(\frac{1}{5}, 0\right)$ (2) $\left(\frac{1}{5}, -1\right)$

(3) $\left(-\frac{1}{5}, 0\right)$ (4) $\left(-\frac{1}{5}, 1\right)$

Solution

(1) We have

$$I_n = \int \tan^n x\, dx$$

$$I_4 + I_6 = \int (\tan^4 x + \tan^6 x)dx$$

$$= \int \tan^4 x(1 + \tan^2 x)dx$$

$$= \int \tan^4 x \cdot \sec^2 x dx$$

Substituting $t = \tan x$, we get

$$I_4 + I_6 = \int t^4 \cdot dt = \frac{t^5}{5} + c$$

That is, $I_4 + I_6 = \frac{\tan^5 x}{5} + c$

On comparison, we get $a = \frac{1}{5}, b = 0$.

15. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1 + \cos x}$ is equal to:

(1) 2 (2) 4

(3) −1 (4) −2

Solution

(1) The given integral is

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$$

That is, $$I=\int \frac{dx}{1+\cos(\pi-x)}$$

By using $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$, we get

$$2I=\int_{\pi/4}^{3\pi/4}\left(\frac{1}{1+\cos x}+\frac{1}{1-\cos x}\right)dx$$

$$=\int_{\pi/4}^{3\pi/4}\left(\frac{2}{1-\cos^2 x}\right)dx$$

$$2I=2\int_{\pi/4}^{3\pi/4}\operatorname{cosec}^2 x dx$$

Therefore, $$I=2\int_{\pi/4}^{3\pi/4}\operatorname{cosec}^2 x dx$$

That is, $$I=(-\cot x)\Big|_{\pi/4}^{3\pi/4}=-(-1-1)=2$$

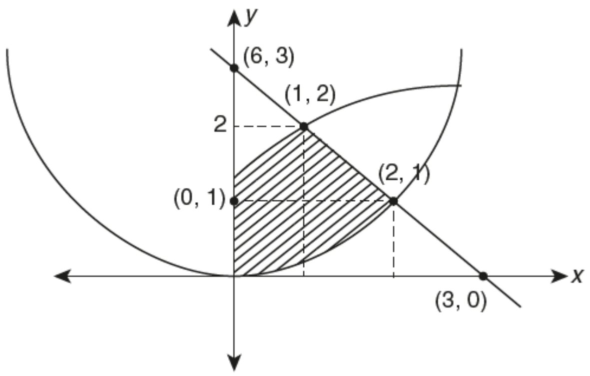
16. The area (in sq.units) of the region $\{(x, y) : x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1+\sqrt{x}\}$ is:

(1) $\frac{3}{2}$ (2) $\frac{7}{3}$

(3) $\frac{5}{2}$ (4) $\frac{59}{12}$

Solution

(3) The given situation is depicted in the following graph:



The area of the given region is

$$\int_0^1\left(1+\sqrt{x}-\frac{x^2}{4}\right)dx+\int_1^3\left(3-x-\frac{x^2}{4}\right)dx$$

$$=\left[x+\frac{x^{3/2}}{3/2}-\frac{x^3}{12}\right]_0^1+\left[3x-\frac{x^2}{2}-\frac{x^3}{12}\right]_1^2$$

$$=\frac{19}{12}+\frac{11}{12}=\frac{5}{2}$$

17. If $(2+\sin x)\frac{dy}{dx}+(y+1)\cos x=0$ and $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to:

(1) $-\frac{2}{3}$ (2) $-\frac{1}{3}$

(3) $\frac{4}{3}$ (4) $\frac{1}{3}$

Solution

(4) It is given that

$$(2+\sin x)\frac{dy}{dx}+(y+1)\cos x=0$$

That is,
$$\frac{dy}{dx}=\frac{-(y+1)\cos x}{2+\sin x}$$

$$\int\frac{dy}{y+1}=-\int\left(\frac{\cos x}{2+\sin x}\right)dx$$

$$\log(y+1)=-\log(2+\sin x)+\log c$$

$$y+1=\frac{c}{2+\sin x} \qquad (1)$$

Given that $y(0) = 1$. Therefore,

$$1+1=\frac{c}{2}\Rightarrow c=4$$

Therefore, the equation of the curve is

$$y+1=\frac{4}{2+\sin x}$$

At $x=\frac{\pi}{2}$, we get

$$y+1=\frac{y}{2+1}$$

$$y=\frac{4}{3}-1=\frac{1}{3}$$

18. Let k be an integer such that the triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then, the orthocentre of this triangle is at the point:

(1) $\left(1,\frac{3}{4}\right)$ (2) $\left(1,-\frac{3}{4}\right)$

(3) $\left(2,\frac{1}{2}\right)$ (4) $\left(2,-\frac{1}{2}\right)$

Solution

(3) We can write the given vertices of the triangle in the following form:

$$\frac{1}{2}\begin{vmatrix} k & -3k & 1 \\ 5 & k & 1 \\ -k & 2 & 1 \end{vmatrix}=28$$

That is, $5k^2 + 13k - 46 = 0$

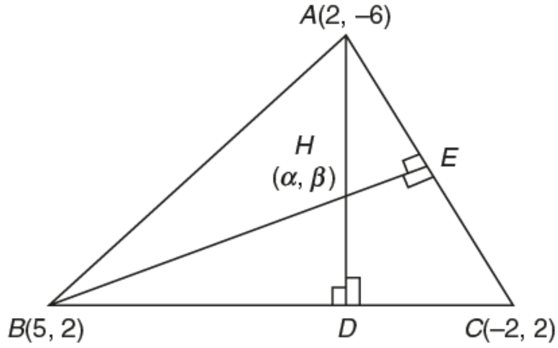
or $5k^2 + 13k + 66 = 0$

From the above, we confirm that no real solution exists. Therefore,

$$k=\frac{-23}{5} \text{ or } k=2$$

Since it is given that k is an integer, we consider only $k = 2$.

Therefore, the vertices are obtained as (2, –6), (5, 2) and (–2, 2) as depicted in the following figure.



Hence, solving the equations of two altitudes, the orthocentre of the triangle is obtained as $\left(2, \frac{1}{2}\right)$.

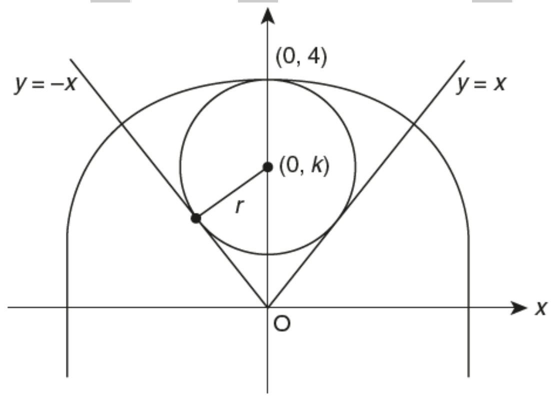
19. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is

(1) $2(\sqrt{2}-1)$ (2) $4(\sqrt{2}-1)$

(3) $4(\sqrt{2}+1)$ (4) $2(\sqrt{2}+1)$

Solution

(2) From the following figure, which depicts the given situation, the circle touches the line. Also, from the graph, the radius is obtained as $4 - k$.



Perpendicular distance from the centre = Radius of the circle

$$4-k = \left|\frac{0-k}{\sqrt{2}}\right|$$

$$16+k^2-8k = \frac{k^2}{2}$$

$$k^2 - 16k + 32 = 0$$

The solution of this quadratic equation is

$$k = \frac{16 \pm \sqrt{256-4(32)}}{2} = \frac{16 \pm 8\sqrt{2}}{2}$$

That is, $k = 8 \pm 4\sqrt{2}$.

Now, we consider $k = 8 - 4\sqrt{2}$ (k should be $0 < k < 4$).
Therefore, the radius of the circle is

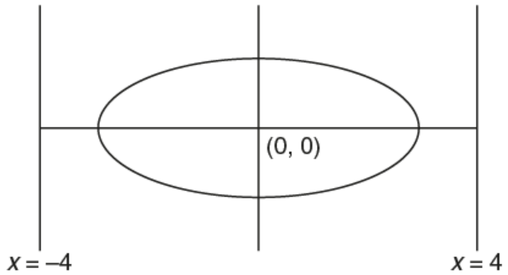
$$4 - k = 4 - (8 - 4\sqrt{2}) = 4(\sqrt{2} - 1)$$

20. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directrices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is:

(1) $4x - 2y = 1$ (2) $4x + 2y = 7$
(3) $x + 2y = 4$ (4) $2y - x = 2$

Solution

(1) The given ellipse is depicted in the following figure:



Eccentricity of ellipse is 1/2.

Now, $$\frac{-a}{e} = -4$$

That is, $$a = 4 \times \frac{1}{2} = 2$$

Therefore, $$b^2 = a^2(1 - e^2) = a^2\left(1 - \frac{1}{4}\right) = 3$$

Hence, the equation of the given ellipse is written as

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$\frac{x}{2} + \frac{2y}{3} \times y' = 0$$

$$y' = \frac{-3x}{4y}$$

$$y'\big|_{(1,3/2)} = \frac{-3}{4} \times \frac{2}{3} = \frac{-1}{2}$$

Therefore, the equation of normal at $\left(1, \frac{3}{2}\right)$ is

$$y - \frac{3}{2} = 2(x - 1)$$

$$2y - 3 = 4x - 4$$

$$\Rightarrow 4x - 2y = 1$$

Answer (1)

21. A hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at (±2, 0). Then the tangent to this hyperbola at P also passes through the point:

(1) $(2\sqrt{2}, 3\sqrt{3})$ (2) $(\sqrt{3}, \sqrt{2})$

(3) $(-\sqrt{2}, -\sqrt{3})$ (4) $(3\sqrt{2}, 2\sqrt{3})$

Solution

(1) Equation of hyperbola is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (1)$$

Foci is (±2, 0); hence,

$$ae = 2$$
$$a^2e^2 = 4$$
$$b^2 = a^2(e^2 - 1)$$

Therefore, $$a^2 + b^2 = 4 \quad (2)$$

Hyperbola passes through $(\sqrt{2}, \sqrt{3})$. Therefore,

$$\frac{2}{a^2} - \frac{3}{9^2} = 1 \quad (3)$$

On solving Eqs. (2) and (3), we get

$a^2 = 8$ (which is rejected) and $a^2 = 1$;

$b^2 = 3$

Now, from Eq. (1), we get the equation of hyperbola as

$$\frac{x^2}{1} - \frac{y^2}{3} = 1$$

Therefore, the equation of tangent is

$$\frac{\sqrt{2}x}{1} - \frac{\sqrt{3}y}{3} = 1$$

That is, this tangent to the hyperbola at P passes through the point $(2\sqrt{2},\ 3\sqrt{3})$.

22. The distance of the point (1, 3, −7) from the plane passing through the point (1, −1, −1), having normal perpendicular to both the lines $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$, is:

(1) $\frac{10}{\sqrt{83}}$ (2) $\frac{5}{\sqrt{83}}$

(3) $\frac{10}{\sqrt{74}}$ (4) $\frac{20}{\sqrt{74}}$

Solution

(1) We have

$$\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix} = 5\hat{i} - \hat{j}(-7) + 3\hat{k}$$

Equation of the given plane is

$$5(x-1)+7(y-1)+3(z+1)=0$$
$$5x+7y+3z+5=0$$

Therefore, the perpendicular distance of plane from the point (1, 3, −7) is

$$\frac{|5+21-21+5|}{\sqrt{25+49+9}}=\frac{10}{\sqrt{83}}$$

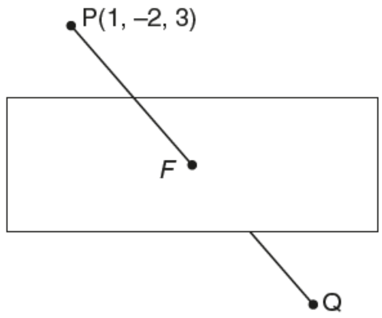
23. If the image of the point P(1, −2, 3) in the plane, $2x+3y-4z+22=0$ measured parallel to the line, $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ is Q, then PQ is equal to:

(1) $2\sqrt{42}$ (2) $\sqrt{42}$

(3) $6\sqrt{5}$ (4) $3\sqrt{5}$

Solution

(1) The given situation is depicted in the following figure:



Line $PQ:\frac{x-1}{1}=\frac{y+2}{4}=\frac{z-3}{5}$

Let us consider $(\lambda+1, 4\lambda-2, 5\lambda+3)$ and point F lies on the plane. Therefore,

$$2(\lambda+1)+3(4\lambda-2)-4(5\lambda+3)+22=0$$
$$-6\lambda+6=0 \Rightarrow \lambda=1$$

That is, point F is obtained as F(2, 2, 8).

Therefore, $PQ=2PF=2\sqrt{42}.$

24. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+j.$ Let $\vec{c}$ be a vector such that $|\vec{c}-\vec{a}|=3,$ $|(\vec{a}\times\vec{b})\times\vec{c}|=3$ and the angle between $\vec{c}$ and $\vec{a}\times\vec{b}$ be 30°. Then $\vec{a}\cdot\vec{c}$ is equal to:

(1) 2 (2) 5

(3) $\frac{1}{8}$ (4) $\frac{25}{8}$

Solution

(1) Let us find the value of $|\vec{a}\times\vec{b}|$:

$$|\vec{a}\times\vec{b}|=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -2\\ 1 & 1 & 0\end{vmatrix}=2\hat{i}-2\hat{j}+\hat{k}=3$$

Now, it is given that $|(\vec{a}\times\vec{b})\times\vec{c}|=3$.

That is, $|\vec{a}\times\vec{b}||\vec{c}|\sin 30° = 3 \; = 3|\vec{c}|\cdot\frac{1}{2} = 3 = |\vec{c}| = 2$

Now, $|\vec{c}-\vec{a}| = 3$

Therefore, $|\vec{c}|^2 + |\vec{a}|^2 - 2\vec{a}\cdot\vec{c} = 9$

$$4+9-2\vec{a}\cdot\vec{c} = 9$$

Therefore, $\vec{a}\cdot\vec{c} = 2$.

Answer (1)

25. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn, one-by-one, with replacement, then the variance of the number of green balls drawn is:

(1) 6 (2) 4

(3) $\frac{6}{25}$ (4) $\frac{12}{5}$

Solution

(4) Total number of balls = 25 (15 balls are in box + 10 yellow balls).

The variance is

$$\sigma^2 = npq$$

Let n be the total number of trials, p be the probability of happening and q be the probability of not happening:

$$n = 10;\ \ p = \frac{15}{25} = \frac{3}{5};\ q = \frac{10}{25} = \frac{2}{5}$$

Thus, the variance of the number of green balls drawn is obtained as follows:

$$\sigma^2 = 10\times\frac{3}{5}\times\frac{2}{5} = \frac{60}{25} = \frac{12}{5}$$

26. For three events A, B and C

P(Exactly one of A or B occurs) = P(Exactly one of B or C occurs)

P(Exactly one of C or A occurs) $= \frac{1}{4}$ and

P(All the three events occur simultaneously) $= \frac{1}{16}$.

Then the probability that at least one of the events occurs, is:

(1) $\frac{7}{16}$ (2) $\frac{7}{64}$

(3) $\frac{3}{16}$ (4) $\frac{7}{32}$

Solution

(1) We have

$$P(\text{Exactly one of A or B occurs}) = P(\text{A}) + P(\text{B}) - 2P(\text{A}\cap\text{B}) = \frac{1}{4}$$

$$P(\text{Exactly one of } B \text{ or } C \text{ occurs}) = P(\text{B}) + P(\text{C}) - 2P(\text{B}\cap\text{C}) = \frac{1}{4}$$

$$P(\text{Exactly one of C or A occurs}) = P(\text{C}) + P(\text{A}) - 2P(\text{C}\cap\text{A}) = \frac{1}{4}$$

Adding all, we get

$$2\Sigma P(\text{A})-2\Sigma P(\text{A}\cap\text{B})=\frac{3}{4}$$

Therefore, $$\Sigma P(A)-\Sigma P(A\cap B)=\frac{3}{8}$$

Now, it is given that all the three events occur simultaneously, which is given by

$$P(\text{A}\cap\text{B}\cap\text{C})=\frac{1}{16}$$

Therefore, the probability that at least one of the events occurs, is

$$P(\text{A}\cup\text{B}\cup\text{C})=\Sigma P(\text{A})-\Sigma P(\text{A}\cap\text{B})+P(\text{A}\cap\text{B}\cap\text{C})$$

$$=\frac{3}{8}+\frac{1}{16}=\frac{7}{16}$$

27. If two different numbers are taken from the set {0, 1, 2, 3, …, 10}; then, the probability that their sum as well as absolute difference are both multiple of 4, is:

(1) $\frac{12}{55}$ (2) $\frac{14}{45}$

(3) $\frac{7}{55}$ (4) $\frac{6}{55}$

Solution

(4) We have

$$n(s) = {}^{11}C_2 = 55$$

Now, the favourable events are as follows:

$$\begin{bmatrix}(0,4)\\(0,8)\\(2,6),(2,10)\\(4,8),(6,10)\end{bmatrix}$$

So, the probability that the sum as well as absolute difference, which are both multiple of 4 is

$$\frac{\text{Favarouble events}}{\text{Total events}}=\frac{6}{55}$$

28. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is:

(1) $\frac{1}{3}$ (2) $\frac{2}{9}$

(3) $-\frac{7}{9}$ (4) $-\frac{3}{5}$

Solution

(3) Let $\cos^2 x = t$. Therefore, from the given equation, we get

$$5\left[\frac{1-t}{t}-t\right]=2(2t-1)+9$$

$$5(1 - t - t^2) = t(4t + 7)$$

$$9t^2 + 12t - 5 = 0$$

$$9t^2 + 15t - 3t - 5 = 0$$

$$(3t - 1)(3t + 5) = 0$$

Thus, we consider $t = \frac{1}{3}$ since $t \neq \frac{-5}{3}$.

Therefore, $$\cos 2x = 2\left(\frac{1}{3}\right) - 1 = \frac{-1}{3}$$

and $$\cos 4x = 2\left(\frac{-1}{3}\right)^2 - 1 = \frac{-7}{9}$$

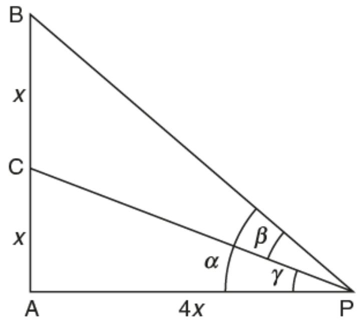
29. Let a vertical tower AB have its end a on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP = 2AB. If $\angle BPC = \beta$, then tanβ is equal to:

(1) $\frac{1}{4}$ (2) $\frac{2}{9}$

(3) $\frac{4}{9}$ (4) $\frac{6}{7}$

Solution

(2) The given situation is depicted as shown in the following figure:



From the figure, we have

$$\beta = \alpha - \gamma$$

Therefore, $$\tan\beta = \frac{\tan\alpha - \tan\gamma}{1 + (\tan\alpha)(\tan\gamma)} = \frac{(1/2) - (1/4)}{1 + (1/8)} = \frac{2}{9}$$

30. The following statement $(p \to q) \to [\sim p \to q) \to q]$ is:

(1) equivalent to $\sim p \to q$.

(2) equivalent to $p \to \sim q$.

(3) a fallacy.

(4) a tautology.

Solution

(4) See the following table for the given statement:

q	***p***	***~p***	**(*p*→*q*)**	**(~*p*→*q*)**	**(~*p*→*q*)→*q***	**(*p*→*q*)→[(~*p*→*q*)→*q*]**
T	T	F	T	T	T	T
F	T	F	F	T	F	T
T	F	T	T	T	T	T
F	F	T	T	F	T	T

From this table, we can confirm that the given statement $(p \to q) \to [\sim p \to q) \to q]$ is a tautology.

