JEE-Main 2018 (Offline) – Mathematics

(Code C)

Chapter: Integrals

Topic: Methods of Integration

1. The integral $\int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx$ is equal to

(1) $-\frac{1}{3(1+\tan^3 x)} + C$ (2) $\frac{1}{1+\cot^3 x} + C$

(3) $-\frac{1}{1+\cot^3 x} + C$ (4) $-\frac{1}{3(1+\tan^3 x)} + C$

(where C is a constant of integration)

Solution: (1) Let us consider

$$I = \int \frac{\sin^2 x \cos^2 x}{[(\sin^2 x + \cos^2 x)(\sin^3 x + \cos^3 x)]^2} dx = \int \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$$

$$= \int \frac{\cos^4 x \left(\frac{\sin^2 x}{\cos^2 x}\right)}{\cos^6 x(\sin^3 x + \cos^3 x)^2} dx = \int \frac{\cos^4 x \left(\frac{\sin^2 x}{\cos^2 x}\right)}{\cos^6 x(\sin^3 x + \cos^3 x)^2} dx$$

$$= \frac{\tan^2 x \sec^2 x}{(1+\tan^3 x)^2} dx$$

Let us substitute as

$$1 + \tan^3 x = t$$

Therefore,

$$3\tan^2 x \sec^2 x dx = dt$$

$$\tan^2 x \sec^2 x dx = \frac{1}{3} dt$$

Thus,

$$I = \frac{1}{3}\int \frac{1}{t^2} dt = \frac{1}{3}\left(-\frac{1}{t}\right) + C$$

$$= -\frac{1}{3(1+\tan^3 x)} + C$$

Chapter: Conic Sections

Topic: Hyperbola

2. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If these tangents intersect at the point T(0, 3) then the area (in sq. units) of ΔPTQ is

(1) $54\sqrt{3}$ (2) $60\sqrt{3}$

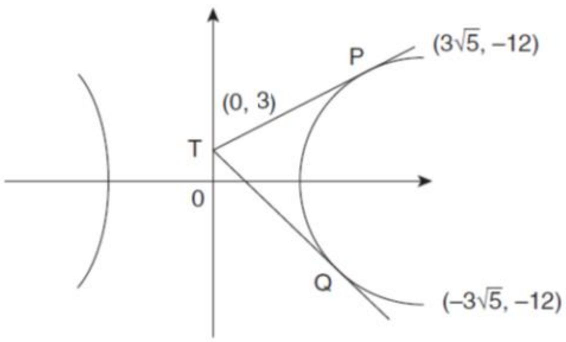
(3) $36\sqrt{5}$ (4) $45\sqrt{5}$

Solution

(4) Given equation can be rewrite as

$$\frac{x^2}{9}-\frac{y^2}{36}=1$$

Now plotting the graph according to given data.



$$\frac{xx_1}{9}-\frac{yy_1}{36}=1$$

Now, $x_1 = 0$ and $y_1 = 3$

That is, $$\frac{0\times x}{9}-\frac{3y}{36}=1$$

$$\frac{-y}{12}=1 \Rightarrow y=-12$$

Also, $$\frac{x^2}{9}-\frac{144}{36}=1$$

$$\frac{x^2}{9}=\frac{180}{36} \Rightarrow x=\pm 3\sqrt{5}$$

So, $$\text{P}=(3\sqrt{5},-12) \text{ and } \text{Q}=(-3\sqrt{5},-12)$$

Therefore, the area of ΔPTQ is

$$A=\frac{1}{2}\begin{vmatrix} 0 & 3 & 1 \\ 3\sqrt{5} & -12 & 1 \\ -3\sqrt{5} & -12 & 1 \end{vmatrix}$$

$$=\frac{1}{2}(-3(6\sqrt{5})-36\sqrt{5}-36\sqrt{5})$$

$$=\frac{1}{2}(-18\sqrt{5}-36\sqrt{5}-36\sqrt{5}) \Rightarrow 45\sqrt{5} \text{ sq. units}$$

Chapter: Conic Sections

Topic: Parabola

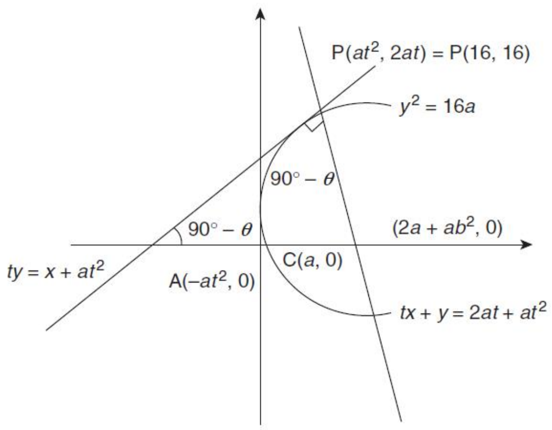
3. Tangent and normal are drawn at P(16, 16) on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle \text{CPB} = \theta$, then a value of $\tan\theta$ is

(1) 2 (2) 3

(3) $\frac{4}{3}$ (4) $\frac{1}{2}$

Solution

(1) From the given data, we plot the graph



From above graph we can say

$$\angle CPB = \theta$$

$$\angle APC = 90 - \theta$$

$$\Rightarrow \angle PAC = 90 - \theta$$

We know that the general equation of the tangent is

$$y^2 = 4ax$$

So, $16x = 4ax \Rightarrow a = 4$

And $2at = 16 \Rightarrow 2 \times 4 \times t = 16 \Rightarrow t = 2$

Now, the slope of the tangent is

$$\tan(90° - \theta) = \frac{1}{t} = \frac{1}{2}$$

$$\cot\theta = \frac{1}{2}$$

Therefore, $\tan\theta = 2$.

Chapter: Vector Algebra

Topic: Product of Two Vectors

4. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u} \cdot \vec{b}$ = 24, then $|\vec{u}|^2$ is equal to

(1) 315 (2) 256

(3) 84 (4) 336

Solution

(4) Let $\vec{u} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{u} \cdot \vec{a} = 0$$

Therefore, $2x + 3y - z = 0$ (1)

Given: $\vec{u} \cdot \vec{b} = 24$

$$y + z = 24 \quad (2)$$

Since, all three given vectors are coplanar

Therefore, $|\vec{u} \quad \vec{a} \quad \vec{b}| = 0$

That is,
$$\begin{vmatrix} x & y & z \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4x - 2y + 2z = 0$$

$$2x - y + z = 0 \quad (3)$$

Now, Eq. (2) + Eq. (3), we have

$$2x + 2z = 24$$

$$x + z = 12 \quad (4)$$

Now, put value of y from Eq. (2) and value of x from Eq. (4) in Eq. (1), we have

$$24 - 2z + 3\,(24 - z) - z = 0$$

That is, $96 = 6z$

$$\Rightarrow z = 16$$

and so, $x = -4$ and $y = 8$. Therefore,

$$\vec{u} = -4\hat{i} + 8\hat{j} + 16\hat{k}$$

$$|\vec{u}|^2 = 16 + 64 + 256 \Rightarrow 336$$

Chapter: Complex Numbers and Quadratic Equations

Topic: Quadratic Equations

5. If $\alpha, \beta \in C$ are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to

(1) 0 (2) 1

(3) 2 (4) –1

Solution

(2) Given

$$x^2 - x + 1 = 0$$

Roots of the equation are

$$x = \frac{1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$x = \frac{1 \pm i\sqrt{3}}{2}$$

So, $$\alpha = \frac{1 + i\sqrt{3}}{2} = -\omega, \beta = \frac{1 - i\sqrt{3}}{2} = -\omega^2$$

Therefore, $$\alpha^{101} + \beta^{107} = (-\omega)^{101} + (-\omega^2)^{107}$$

$$= -\omega^{101} - \omega^{214}$$

$$= -\omega^{99+2} - \omega^{213+1} = -\omega^{3\times 11}\omega^2 - \omega^{3\times 71}\omega$$

Since, $\omega^3 = 1, 1 + \omega + \omega^2 = 0$

$$= -\omega^2 - \omega = 1$$

Chapter: Application of Integrals

Topic: Area between Two Curves

6. Let $g(x)=\cos x^2$, $f(x)=\sqrt{x}$ and $\alpha,\beta(\alpha<\beta)$ be the roots of the quadratic equation $18x^2-9\pi x+\pi^2=0$. Then, the area (in sq. units) bounded by the curve $y=(g\circ f)(x)$ and the lines $x=\alpha$, $x=\beta$ and $y=0$ is

(1) $\frac{1}{2}(\sqrt{3}+1)$ (2) $\frac{1}{2}(\sqrt{3}+\sqrt{2})$

(3) $\frac{1}{2}(\sqrt{2}-1)$ (4) $\frac{1}{2}(\sqrt{3}-1)$

Solution

(4) Given

$$g(x)=\cos x^2 \text{ and } f(x)=\sqrt{x}$$

$$18x^2-9\pi x+\pi^2=0$$

$$18x^2-6\pi x-3\pi x+\pi^2=0$$

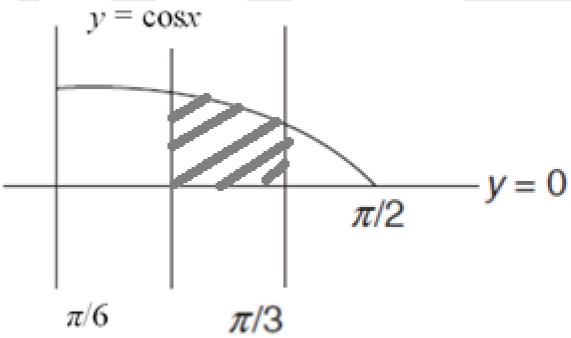
$$(6x-\pi)(3x+\pi)=0$$

So, $\alpha=(6x-\pi),\beta=(3x+\pi)$

Now, $6x-\pi=0,\ 3x+\pi=0$

$$x=\frac{\pi}{6},\frac{\pi}{3}$$

$$y=(g\circ f)(x)=\cos x$$



Now, the area (in sq. units) bounded by the given curve is

$$A=\int_{\pi/6}^{\pi/3}\cos x\,dx=(\sin x)\Big|_{\pi/6}^{\pi/3}$$

$$=\frac{\sqrt{3}}{2}-\frac{1}{2}=\frac{1}{2}(\sqrt{3}-1)$$

Chapter: Binomial Theorem

Topic: Binomial Theorem for Positive Integral Indices

7. The sum of the coefficients of all odd degree terms in the expansion of $(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5,(x>1)$ is

(1) 0 (2) 1

(3) 2 (4) –1

Solution

(3) Given: $(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5$

Let the given equation expand as,

$$(T_1 + T_2 + T_3 + T_4 + T_5 + T_6) + (T_1 - T_2 + T_3 - T_4 + T_5 - T_6)$$
$$= 2(T_1 + T_3 + T_5)$$
$$= 2({}^5C_0(x)^5 + {}^5C_2(x)^3\left(\sqrt{x^3-1}\right)^2 + {}^5C_4(x)^1\left(\sqrt{x^3-1}\right)^4$$
$$= 2(x^5 + 10x^3(x^3 - 1) + 5x(x^6 + 1 - 2x^3)$$
$$= 2(x^5 + 10x^6 - 10x^3 + 5x^7 + 5x - 10x^4)$$
$$= 2(5x^7 + 10x^6 + x^5 - 10x^4 - 10x^3 + 5x)$$
$$= (10x^7 + 20x^6 + 2x^5 - 10x^4 - 20x^3 + 10x)$$

The sum of odd degree terms

$$10 + 2 - 20 + 10 = 2$$

Chapter: Sequences and Series

Topic: Arithmetic Progression (A.P.)

8. Let $a_1, a_2, a_3, \ldots, a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \cdots + a_{17}^2 = 140m$, then m is equal to

(1) 68 (2) 34

(3) 33 (4) 66

Solution

(2) Given

$$a_1 + a_5 + a_9 + a_{13} + \cdots + a_{49} = 416$$

That is, $a_1 + (a_1 + 4d) + (a_1 + 8d) + (a_1 + 12d) + \cdots + (a_1 + 48d) = 416$

$$13a_1 + 4d(1 + 2 + 3 + \cdots + 12) = 416$$
$$13a_1 + 4d \times \frac{12\times 13}{2} = 416$$
$$13a_1 + 24 \times 13d = 416$$
$$a_1 + 24d = 32 \quad (1)$$

We have, $a_9 + a_{43} = 66$

$$a_1 + 8d + a_1 + 42d = 66$$
$$2a_1 + 50d = 66$$
$$a_1 + 25d = 33 \quad (2)$$

Solving, using Eq. (1) and Eq. (2). We get $d = 1$. Therefore, $a_1 = 8$.

$$a_1^2 + (a_1 + d)^2 + (a_1 + 2d)^2 + \cdots + (a_1 + 16d)^2 = 140m$$
$$17a_1^2 + d^2(1^2 + 2^2 + \cdots + 16)^2 + 2a_1 d(1 + 2 + 3 + \cdots + 16) = 140m$$
$$17\times 64 + \frac{16\times 17\times 33}{6} + 2\times 8\times 1\times \frac{16\times 17}{2} = 140m$$
$$17 \times 64 + 8 \times 11 \times 17 + 8 \times 16 \times 17 = 140m$$
$$17 \times 16 + 22 \times 17 + 2 \times 16 \times 17 = 35m$$
$$272 + 374 + 544 = 35m$$
$$1190 = 35m$$

Therefore, $m = 34$.

Chapter: Statistics

Topic: Mean Deviation

9. If $\sum_{i=1}^{9}(x_i - 5) = 9$ and $\sum_{i=1}^{9}(x_i - 5)^2 = 45$, then the standard deviation of the nine items $x_1, x_2, \ldots, x_9$ is

(1) 4 (2) 2

(3) 3 (4) 9

Solution

(2) Standard deviation (SD) is independent of shifting of origin. Therefore, the standard deviation of the nine items is

$$\text{SD} = +\sqrt{\text{var}(x_i - 5)}$$

Given: $\sum_{i=1}^{9}(x_i - 5) = 9$ and $\sum_{i=1}^{9}(x_i - 5)^2 = 45$

$$\sum_{i=1}^{9} x_i - 45 = 9 \Rightarrow \sum_{i=1}^{9} x_i = 54$$

$$\sum_{i=1}^{9}(x_i^2 + 25 - 10x_1) = 45 = \sum_{i=1}^{9} x_i^2 - 10\sum_{i=1}^{9} x_1 + 225 = 45$$

$$\Rightarrow \sum_{i=1}^{9} x_i^2 = 360$$

Variation is given by

$$\text{var}(x) = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

Therefore standard deviation is given as

$$\text{SD} = \sqrt{\frac{360}{9} - \left(\frac{54}{9}\right)^2} \Rightarrow \sqrt{40 - 36} \Rightarrow \sqrt{4} = 2$$

Chapter: Trigonometric Functions

Topic: Trigonometric Functions of Sum and Difference of Two Angles

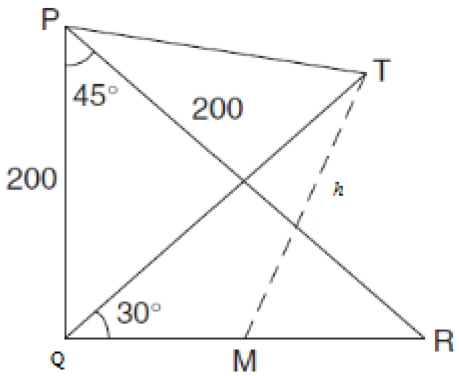
10. PQR is a triangular park with PQ = PR = 200 m. A T.V. tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P, Q and R are respectively, 45°, 30° and 30°, then the height of the tower (in m) is

(1) 50 (2) $100\sqrt{3}$

(3) $50\sqrt{3}$ (4) 100

Solution

(4) Let height of the tower be TM = h; QM = MR = x



From figure we can say that

$$\text{PM} = \sqrt{40000 - x^2}$$

$$\tan 45° = \frac{\text{TM}}{\text{PM}} = \frac{h}{\sqrt{40000 - x^2}}$$

$$h^2 = 40000 - x^2$$

$$h^2 + x^2 = 40000 \qquad (1)$$

Therefore,

$$\tan 30° = \frac{\text{TM}}{\text{QM}}$$

$$x = \sqrt{3}h \qquad (2)$$

From Eqs. (1) and (2), we have

$$4h^2 = 40000$$

$$h = 100 \text{ m}$$

Chapter: Sets

Topic: Practical Problems on Union and Intersection of Two Sets

11. Two sets A and B are as under:

$$A = \{(a, b) \in \mathbb{R} \times \mathbb{R} : |a - 5| < 1 \text{ and } |b - 5| < 1\};$$
$$B = \{(a, b) \in \mathbb{R} \times \mathbb{R} : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}$$

Then

(1) $A \subset B$

(2) $A \cap B = \phi$ (an empty set)

(3) neither $A \subset B$ nor $B \subset A$

(4) $B \subset A$

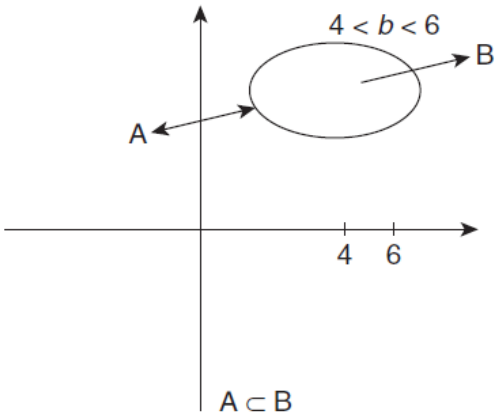
Solution

(1) We have

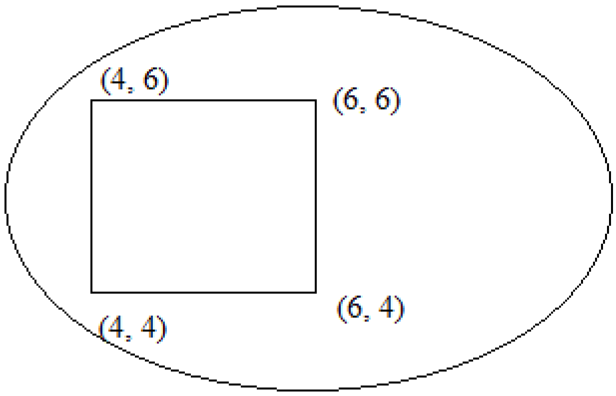
$$-1 < a - 5 < 1$$

$$4 < a < 6$$

$$4 < b < 6$$



$$\frac{(a-6)^2}{3^2}+\frac{(b-5)^2}{2^2}=1$$



The ellipse passes through (4, 6); therefore,

$$\frac{16+9-36}{36}=\frac{25-36}{36}<0$$

That is, $A \subset B$.

Chapter: Permutations and Combinations

Topic: Combinations

12. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. The number of such arrangements is

(1) less than 500
(2) at least 500 but less than 750
(3) at least 750 but less than 1000
(4) at least 1000

Solution

(4) The number of ways the novels arranged is

$$x = {}^6C_4 \times {}^3C_1 \times 4!$$
$$= 15 \times 3 \times 24$$

$= 1080$

Chapter:

Topic:

13. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in \mathbb{R} - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is

(1) -3 (2) $-2\sqrt{2}$

(3) $2\sqrt{2}$ (4) 3

Solution

(3) Given

$$f(x) = x^2 + \frac{1}{x^2} \text{ and } g(x) = x - \frac{1}{x}$$

Therefore,
$$h(x) = \frac{f(x)}{g(x)} \Rightarrow \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} \Rightarrow \frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)}$$

For $x - \frac{1}{x} < 0$,
$$\frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)} \le -2\sqrt{2}$$

Therefore, $-2\sqrt{2}$ is the local maximum value.

For $x - \frac{1}{x} > 0$,
$$\frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)} \ge 2\sqrt{2}$$

Therefore, $2\sqrt{2}$ is the local minimum value.

Chapter: Limits and Derivatives

Topic: Limits

14. For each $t \in \mathbb{R}$, let $[t]$ be the greatest integer less than or equal to t. Then

$$\lim_{x \to 0+} x\left[\left(\frac{1}{x}\right) + \left(\frac{2}{x}\right) + \cdots + \left(\frac{15}{x}\right)\right]$$

(1) is equal to 15.
(2) is equal to 120.
(3) does not exist (in $\mathbb{R}$).
(4) is equal to 0.

Solution

(2) Given

$$\lim_{x \to 0^+} x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \cdots + \left[\frac{15}{x}\right]\right)$$

$$\lim_{x \to 0^+} \left(x\left[\frac{1}{x}\right] + x\left[\frac{2}{x}\right] + \cdots + x\left[\frac{15}{x}\right]\right) = 1 + 2 + 3 + \cdots + 15$$

$$= \frac{15}{2}(15+1) = 120$$

Chapter: Integrals

Topic: Definite Integral

15. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$ is

(1) $\frac{\pi}{2}$ (2) 4π

(3) $\frac{\pi}{4}$ (4) $\frac{\pi}{8}$

Solution

(3) Given

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx = \int_{0}^{\pi/2} \left(\frac{\sin^2 x}{1+2^x} + \frac{\sin^2 x}{1+2^{-x}} \right) dx$$

$$\left[\text{Using property } \int_{-\theta}^{+\theta} f(x)dx = \int_0^{\theta} (f(x) + (-x))dx \right]$$

$$I = \int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \frac{(1-\cos(2x))}{2} dx$$

$$= \frac{1}{2}\left(x - \frac{\sin(2x)}{2} \right)_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{4}$$

Chapter: Probability

Topic: Conditional Probability

16. A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is

(1) $\frac{2}{5}$ (2) $\frac{1}{5}$

(3) $\frac{3}{4}$ (4) $\frac{3}{10}$

Solution

(1) Let R = Red balls, B = Black balls. Therefore,

$$4R + 6B = 10$$

That is,

$$P = \left(\frac{4}{10} \times \frac{6}{12} \right) + \left(\frac{6}{10} \times \frac{4}{12} \right)$$

$$= \frac{24}{120} + \frac{24}{120} = \frac{48}{120} = \frac{2}{5}$$

Chapter: Three Dimensional Geometry

Topic: Direction Cosines and Direction Ratios of a Line

17. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane, $x + y + z = 7$ is

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$

(3) $\sqrt{\frac{2}{3}}$ (4) $\frac{2}{\sqrt{3}}$

Solution

(3) We have the points A(5, −1, 4) and B(4, −1, 3) which join a line segment.

Now, $AB = \sqrt{(4-5)^2 + (-1-(-1))^2 + (3-4)^2} = \sqrt{1+1} = \sqrt{2}$.

The direction ratio of AB is <1, 0, 1>.

Let the angle between line AB and the plane be θ.

$$\sin\theta = \frac{2}{\sqrt{6}}$$

$$\cos\theta = \frac{1}{\sqrt{3}}$$

The projection of line *AB* on the plane is

$$AB\cos\theta = \sqrt{\frac{2}{3}}$$

Chapter: Trigonometric Functions

Topic: Trigonometric Equations

18. If sum of all the solutions of the equation $8\cos x\left[\cos\left(\frac{\pi}{6}+x\right)\cdot\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right] = 1$ in $[0, \pi]$ is $k\pi$, then k is equal to

(1) $\frac{13}{9}$ (2) $\frac{8}{9}$

(3) $\frac{20}{9}$ (4) $\frac{2}{3}$

Solution

(1) Given

$$8\cos x\left[\cos\left(\frac{\pi}{6}+x\right)\cdot\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right] = 1$$

$$8\cos x\left(\left(\cos^2\frac{\pi}{6}-\sin^2 x\right)-\frac{1}{2}\right) = 1$$

$$8\cos x\left(\left(\frac{3}{4}-\sin^2 x\right)-\frac{1}{2}\right) = 1$$

$$6\cos x - 8\cos x(\sin^2 x) - 4\cos x = 1$$

$$6\cos x - 8\cos x(1-\cos^2 x) - 4\cos x - 1 = 0$$

$$8\cos^3 x - 6\cos x - 1 = 0$$

$$2(4\cos^3 x - 3\cos x) = 1$$

$$2(\cos 3x) = 1$$

That is, $\cos 3x = \frac{1}{2}$

Now, $3x = 2n\pi \pm \frac{\pi}{3}$

$$x = (6n \pm 1)\frac{\pi}{9}$$

Now, for $n = 0$: $x = \frac{\pi}{9}$

and for $n = 1$: $x = \frac{7\pi}{9}, \frac{5\pi}{9}$

Now, the sum is

$$S = \frac{13\pi}{9} = k\pi \Rightarrow k = \frac{13}{9}$$

Chapter: Straight Lines

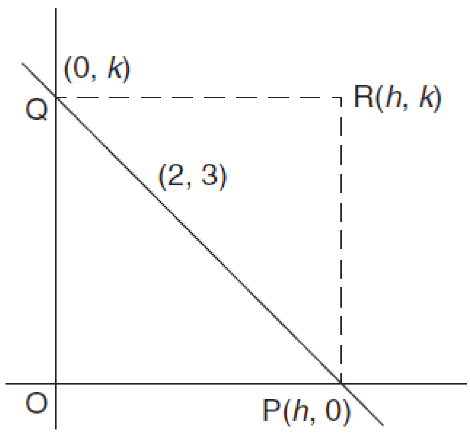
Topic: General Equation of a Line

19. A straight line through a fixed point (2, 3) intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle OPRQ is completed, then the locus of R is

(1) $2x + 3y = xy$ (2) $3x + 2y = xy$
(3) $3x + 2y = 6xy$ (4) $3x + 2y = 6$

Solution

(2) From the given data, we can draw as



From above figure, we can say

$$\begin{vmatrix} 0 & k & 1 \\ 2 & 3 & 1 \\ h & 0 & 1 \end{vmatrix} = 0$$

$$-(2 - h) + (1(-3h)) = 0$$

$$-2y + xy - 3x = 0$$

$$3x + 2y = xy$$

Chapter: Sequences and Series

Topic: Sum to n terms of Special Series

20. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series $1^2 + 2\cdot2^2 + 3^2 + 2\cdot4^2 + 5^2 + 2\cdot6^2 + \cdots$ If $B - 2A = 100\lambda$, then λ is equal to

(1) 248 (2) 464

(3) 496 (4) 232

Solution

(1) The given series is

$$1^2 + 2\cdot2^2 + 3^2 + 2\cdot4^2 + 5^2 + 2\cdot6^2 + \cdots$$

The sum of the first 20 terms is

$$A = 1^2 + 2.2^2 + \cdots + 2.20^2$$
$$= (1^2 + 2^2 + \cdots + 20^2) + (2^2 + 4^2 + \cdots + 20^2)$$
$$= \frac{20\times21\times41}{6} + \frac{4\times10\times11\times21}{6}$$
$$\Rightarrow \frac{20\times21}{6}(41+22) = 70 \times 63 = 4410$$

The sum of the first 40 terms is

$$B = 1^2 + 2\cdot2^2 + \ldots + 2\cdot40^2$$
$$= (1^2 + 2^2 + \cdots + 40^2) + (2^2 + 4^2 + \cdots + 40^2)$$
$$= \frac{40\times41\times81}{6} + \frac{4\times20\times21\times41}{6}$$
$$= \frac{40\times41}{6}(81+42)$$
$$= \frac{40\times41}{6}\times(123)$$
$$\Rightarrow 20(41)^2 = 33620$$

Therefore,

$$B - 2A = 100\lambda \Rightarrow \lambda = \frac{33620-8820}{100} = 248$$

Chapter: Application of Derivatives

Topic: Rate of Change of Quantities

21. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is

(1) $\frac{7}{2}$ (2) 4

(3) $\frac{9}{2}$ (4) 6

Solution

(3) Given: $y^2 = 6x$ and $9x^2 + by^2 = 16$.

Slopes of the given curves is given by

$$2y\frac{dy}{dx} = 6$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{y} \quad (1)$$

And,
$$18x + 2by\frac{dy}{dx} = 0$$

$$9x + by\frac{dy}{dx} = 0$$

$$by\frac{dy}{dx} = -9x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-9x}{by} \quad (2)$$

Both curves intersect each other at right angle, that is slopes of both curves are perpendicular to each other's.

So,

$$\frac{3}{y} \times \frac{-9x}{by} = -1$$

$$-27x = -by^2$$

$$-27x = -b \times 6x$$

Therefore,

$$b = \frac{27}{6} = \frac{9}{2}$$

Chapter:

Topic:

22. Let the orthocentre and centroid of a triangle be A(–3, 5) and B(3, 3), respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is

(1) $2\sqrt{10}$ (2) $3\sqrt{\frac{5}{2}}$

(3) $\frac{3\sqrt{5}}{2}$ (4) $\sqrt{10}$

Solution

(2) From given data we can draw

We know that centroid divides orthocentre and circumcenter in ratio 2:1. So,

$$3 = \frac{2x-3}{3}$$

$$9 = 2x - 3$$

$$12 = 2x \Rightarrow x = 6$$

Also,
$$3 = \frac{2y+5}{3}$$

$$9 = 2y + 5$$

$$2y = 4 \Rightarrow y = 2$$

Therefore, co-ordinates of the centroid is (6, 2)

Now, the radius of the circle is

$$r = \frac{\text{AC}}{2} = \frac{1}{2}\sqrt{(6-(-3))^2 + (2-5)^2} = \frac{1}{2}\sqrt{81+9}$$

$$= \frac{1}{2}\sqrt{90} \quad = \frac{3}{2}\sqrt{10}$$

$$r = 3\sqrt{\frac{5}{2}}$$

Chapter: Continuity and Differentiability

Topic: Exponential and Logarithmic Functions

23. Let $S = \{t \in R : f(x) = |x - \pi| \cdot (e^{|x|} - 1)\sin|x|$ is not differentiable at $t\}$. Then, the set S is equal to

(1) {0} (2) { π }

(3) {0, π } (4) ϕ (an empty set)

Solution

(4) Given

$$f(x) = |x - \pi| \cdot (e^{|x|} - 1)\sin|x|$$

According to the given options, we have to check only at $x = 0$ and at $x = \pi$.

- **At $x = 0$:**

$$\text{L.H.D} = \lim_{h \to 0^+} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0^+} \frac{(\pi + h) \times (e^h - 1)\sin h}{-h}$$

$$\text{R.H.D} = \lim_{h \to 0^+} \frac{(\pi - h)(e^h - 1)\sin h}{h}$$

$$= 0$$

So, differentiable at $x = 0$.

- **At $x = \pi$:**

$$f(\pi) = 0$$

$$\text{L.H.D} = \lim_{h \to 0^+} \frac{f(\pi - h) - f(\pi)}{-h}$$

$$= \lim_{h \to 0^+} \frac{h(e^{\pi - h} - 1)\sin h}{h} = 0$$

$$\text{R.H.D} = \lim_{h \to 0} \frac{f(\pi + h) - f(\pi)}{h}$$

$$= \lim_{h \to 0} \frac{-h(e^{\pi + h} - 1)\sin h}{h} = 0$$

So, differentiable at $x = \pi$.

Hence, the set S is equal to ϕ (an empty set).

Chapter: Determinants

Topic: Applications of Determinants and Matrices

24. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x - A)^2$, then the ordered pair (A, B) is equal to

(1) (–4, 3) (2) (–4, 5)
(3) (4, 5) (4) (–4, –5)

Solution

(2) Given

$$\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

$C_1 \to C_1 + C_2 + C_3$:

$$\begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

$$\Rightarrow \ (5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x-4 & 2x \\ 1 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

$C_2 \to C_2 - C_1$:

$$(5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x-4 & 0 \\ 1 & 0 & -x-4 \end{vmatrix} = (A \times Bx)(x-A)^2$$

$$(5x - 4)\ (x + 4)^2 = (A + Bx)(x - A)^2$$

That is, $A = -4$ and $B = 5$.

Chapter: Mathematical Reasoning

Topic: Statements

25. The Boolean expression $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to

(1) p (2) q
(3) $\sim q$ (4) $\sim p$

Solution

(4) We have

$$\begin{aligned} \sim(p \vee q) \vee (\sim p \wedge q) &= (\sim p \vee \sim q) \vee (\sim p \wedge q) \\ &= \sim p \wedge (\sim q \vee q) \\ &= \sim p \wedge (t) \\ &= \sim p \end{aligned}$$

Chapter: Determinants

Topic: Applications of Determinants and Matrices

26. If the system of linear equations

$x + ky + 3z = 0$
$3x + ky - 2z = 0$
$2x + 4y - 3z = 0$

has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to

(1) 10 (2) – 30
(3) 30 (4) –10

Solution

(1) Given

$$x + ky + 3z = 0,\ 3x + ky - 2z = 0,\ 2x + 4y - 3z = 0$$

For non-zero solutions, we know that

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$$

$$-3k + 8 - k(-9 + 4) + 3(12 - 2k) = 0$$
$$-3k + 8 + 5k + 36 - 6k = 0$$
$$-4k = -44$$
$$k = 11$$

Therefore,
$$x + 11y + 3z = 0$$
$$3x + 11y + 2z = 0$$
$$2x + 4y - 3z = 0$$

Now, $z = t$; therefore,

$$x + 11y = -3t$$
$$3x + 11y = 2t$$
$$2x = 5t$$
$$\Rightarrow x = \frac{5t}{2}$$

Therefore,
$$y = \frac{-3z - x}{11} = \frac{-3t - \frac{5t}{2}}{11}$$
$$= \frac{-11t}{2 \times 11} = -\frac{t}{2}$$

Hence, then $\frac{xz}{y^2}$ is equal to

$$\frac{xz}{y^2} = \frac{\left(\frac{5t \times t}{2}\right)}{\left(\frac{t^2}{4}\right)} \Rightarrow \frac{5}{2} \times 4 \Rightarrow 10$$

Chapter: Sets

Topic: Sets and their representations

27. Let $S = \{x \in \mathbb{R} : x \geq 0 \text{ and } 2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}$. Then, S

(1) contains exactly one element.

(2) contains exactly two elements.
(3) contains exactly four elements.
(4) is an empty set.

Solution

(2) Given

$$2\left|\sqrt{x}-3\right|+\sqrt{x}(\sqrt{x}-6)+6=0$$

- **Case 1**: $\sqrt{x}\geq 3 \Rightarrow x\geq 9$

$$2\sqrt{x}-6+x-6\sqrt{x}+6=0$$
$$x-4\sqrt{x}=0$$
$$\sqrt{x}(\sqrt{x}-4)=0$$
$$\sqrt{x}=0,4 \;\Rightarrow\; x = 0, 16$$

So, $x=16$ is accepted.

- **Case 2:** $\sqrt{x}<3 \Rightarrow x<9$

$$-2\sqrt{x}+6+x-6\sqrt{x}+6=0$$
$$x-8\sqrt{x}+12=0$$
$$(\sqrt{x}-6)(\sqrt{x}-2)=0$$
$$\sqrt{x}=2,6 \Rightarrow x=4,36$$

So, $x=4$ is accepted.

Therefore, S has two elements: $x=4$ and $x=16$.

Chapter: Conic Sections

Topic: Parabola

28. If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$, then the value of c is

(1) 185 (2) 85
(3) 95 (4) 195

Solution

(3) The equation of tangent at (1, 7) for $x^2 = y - 6$ is

$$x=\frac{1}{2}(y+7)-6$$
$$\Rightarrow\; 2x = y + 7 - 12$$
$$\Rightarrow 2x - y + 5 = 0$$

The centre of the circle $x^2 + y^2 + 16x + 12y + c = 0$ is (−8, –6).

Perpendicular from centre (–8, –6) to the tangent $2x - y + 5 = 0$ should be equal to radius of the circle.

Therefore,

$$\left|\frac{-16+6+5}{\sqrt{5}}\right|=\sqrt{64+36-c}$$
$$\sqrt{5}=\sqrt{100-c} \Rightarrow c=95$$

Chapter: Differential Equations

Topic: General and Particular Solutions of a Differential Equation

29. Let $y = y(x)$ be the solution of the differential equation $\sin x \frac{dy}{dx} + y\cos x = 4x,\ x \in (0,\ \pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to

(1) $-\frac{8}{9\sqrt{3}}\pi^2$ (2) $-\frac{8}{9}\pi^2$

(3) $-\frac{4}{9}\pi^2$ (4) $\frac{8}{9\sqrt{3}}\pi^2$

Solution

(2) Given

$$\sin x \frac{dy}{dx} + y\cos x = 4x,\ x \in (0,\ \pi).$$

$$\frac{dy}{dx} + y\cot x = 4x\ \text{cosec}x$$

$$\text{I.F.} = e^{\int \cot x dx} = \sin x$$

So, $$y\sin x = \int 4x\ \text{cosec}\ x \cdot \sin x dx + c$$

$$y\sin x = 2x^2 + c$$

Since, $y\left(\frac{\pi}{2}\right) = 0$, we get

$$0 = 2\left(\frac{\pi}{2}\right)^2 + c$$

$$c = \frac{-\pi^2}{2}$$

So, $$y\sin x = 2x^2 - \frac{\pi^2}{2}$$

Therefore, for $y\left(\frac{\pi}{6}\right)$, we have

$$y\sin\left(\frac{\pi}{6}\right) = 2\left(\frac{\pi}{6}\right)^2 - \frac{\pi^2}{2} \Rightarrow \frac{y}{2} = \pi^2\left(\frac{1}{18} - \frac{1}{2}\right) \Rightarrow y = -\frac{8\pi^2}{9}$$

Chapter: Three Dimensional Geometry

Topic: Distance of a Point from a Plane

30. If L_1 is the line of intersection of the planes $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$ and L_2 is the line of intersection of the planes $x + 2y - z - 3 = 0$, $3x - y + 2z - 1 = 0$, then the distance of the origin from the plane, containing the lines L_1 and L_2 is

(1) $\frac{1}{3\sqrt{2}}$ (2) $\frac{1}{2\sqrt{2}}$

(3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{4\sqrt{2}}$

Solution

(1) Given

$$\text{First two planes: } 2x-2y+3z-2=0, x-y+z+1=0$$

Equation of plane passes through line of intersection of first two planes is

$$2x-2y+3z-2+\lambda(x-y+z+1)=0$$

$$x(\lambda+2)-y(2+\lambda)+z(\lambda+3)+(\lambda-2)=0 \quad (1)$$

The Eq. (1) should have infinite number of solutions with the last two given planes, that are, $x + 2y - z - 3 = 0$, $3x - y + 2z - 1 = 0$.

So,

$$\begin{vmatrix} (\lambda+2) & -(\lambda+2) & (\lambda+3) \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 0$$

On solving, we get $\lambda = 5$

Now put this value of λ in Eq. (1), we have

$$7x-7y+8z+3=0$$

Therefore, perpendicular distance from the origin (0, 0, 0) of the plane is

$$\frac{3}{\sqrt{167}}=\frac{1}{3\sqrt{2}}$$