JEE ADVANCED 2013
PAPER 1
MATHEMATICS

Only One Option Correct Type

This section contains **TEN** questions. Each has **FOUR** options (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

1. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is

(A) $\frac{23}{25}$ (B) $\frac{25}{23}$

(C) $\frac{23}{24}$ (D) $\frac{24}{23}$

Solution

We have

$$\begin{aligned}\cot\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right) &= \cot\sum_{n=1}^{23}\cot^{-1}(1+k(k+1))\\ &= \cot\sum_{n=1}^{23}(\tan^{-1}(k+1)-\tan^{-1}k)\\ &= \cot(\tan^{-1}24-\tan^{-1}1)\\ &= \cot\left[\tan^{-1}\left(\frac{23}{25}\right)\right]\\ &= \frac{25}{23}.\end{aligned}$$

2. Let $\overrightarrow{PR}=3\hat{i}+\hat{j}-2\hat{k}$ and $\overrightarrow{SQ}=\hat{i}-3\hat{j}-4\hat{k}$ determine diagonals of a parallelogram $PQRS$ and $\overrightarrow{PT}=\hat{i}+2\hat{j}+3\hat{k}$ be another vector. Then, the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}, \overrightarrow{PQ}$ and $\overrightarrow{PS}$ is

(A) 5 (B) 20

(C) 10 (D) 30

Solution

Let $\vec{a}$ and $\vec{b}$ be the sides of the parallelogram whose diagonals be $\overrightarrow{PR}$ and $\overrightarrow{SQ}$, as shown in the following figure. Therefore,

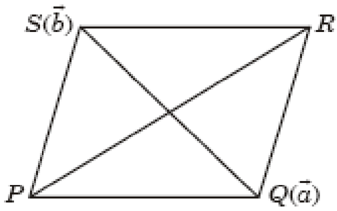
$$\overrightarrow{PR}=\vec{a}+\vec{b}=3\hat{i}+\hat{j}-2\hat{k}$$
$$\overrightarrow{SQ}=\vec{a}-\vec{b}=\hat{i}-3\hat{j}-4\hat{k}$$

These imply that

$$\vec{a}=2\hat{i}-\hat{j}-3\hat{k};$$
$$\vec{b}=\hat{i}+2\hat{j}+\hat{k}.$$

Therefore, the volume of the parallelepiped formed by $\vec{a},\vec{b}$ and $\overrightarrow{PT}$ is

$$\begin{vmatrix}2 & -1 & -3\\ 1 & 2 & 1\\ 1 & 2 & 3\end{vmatrix}=10$$



3. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$

(C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

Solution

As α satisfies $|z - z_0| = r$, we have

$$|\alpha - z_0| = r$$
$$\Rightarrow (\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = r^2$$

Therefore,

$$\alpha\bar{\alpha} - \alpha\bar{z}_0 - \bar{\alpha}z_0 + |z_0|^2 = r^2 \qquad (1)$$

Also, since

$$\frac{1}{\bar{\alpha}} = \frac{\alpha}{|\alpha|^2}$$

satisfies $|z - z_0| = 2r$, we have

$$\left|\frac{1}{\bar{\alpha}} - z_0\right| = 2r$$
$$\Rightarrow \left(\frac{1}{\bar{\alpha}} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) = 4r^2$$
$$\Rightarrow (1 - \bar{\alpha}z_0)(1 - \alpha\bar{z}_0) = 4r^2 \cdot |\alpha|^2$$
$$\Rightarrow 1 - \alpha\bar{z}_0 - \bar{\alpha}z_0 + |\alpha|^2|z_0|^2 = 4r^2|\alpha|^2 \qquad (2)$$

Subtracting Eq. (2) from Eq. (1), we have

$$(|\alpha|^2 - 1) + (1 - |\alpha|^2)|z_0|^2 = (1 - 4|\alpha|^2)r^2$$

That is,

$$(|\alpha|^2 - 1)(1 - |z_0|^2) = r^2(1 - 4|\alpha|^2) \qquad (3)$$

Also, we have

$$2|z_0|^2 = r^2 + 2$$

That is,

$$-2(1 - |z_0|^2) = r^2 \qquad (4)$$

Dividing Eq. (3) by Eq. (4), we get

$$\frac{|\alpha|^2 - 1}{-2} = 1 - 4|\alpha|^2$$
$$|\alpha|^2 - 1 = -2 + 8|\alpha|^2$$
$$7|\alpha|^2 = 1 \Rightarrow |\alpha| = \frac{1}{\sqrt{7}}.$$

4. For $a > b > c > 0$, the distance between (1, 1) and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then

(A) $a + b - c > 0$ (B) $a - b + c < 0$

(C) $a - b + c > 0$ (D) $a + b - c < 0$

Solution

We know that

$$ax + by + c = 0 \quad (1)$$

$$bx + ay + c = 0 \quad (2)$$

Solving, we get

$$x = \frac{-c}{a+b}$$

From Eqs. (1) and (2), we get $y = x$. That is, the point of intersection lies on $y = x$. This implies that

$$y = \frac{-c}{a+b}$$

It is given that

$$\sqrt{\left(1+\frac{c}{a+b}\right)^2 + \left(1+\frac{c}{a+b}\right)^2} < 2\sqrt{2}$$

That is,

$$\sqrt{2}\left(1+\frac{c}{a+b}\right) < 2\sqrt{2}$$

$$\Rightarrow \frac{a+b+c}{a+b} < 2$$

$$\Rightarrow a+b+c < 2a+2b$$

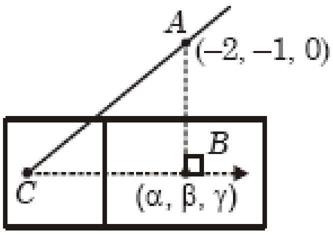
$$\Rightarrow a+b-c > 0$$

5. Perpendiculars are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane $x + y + z = 3$. The feet of perpendiculars lie on the line

(A) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (B) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$

(C) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (D) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

Solution



Let us consider that

$$\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3} = \lambda$$

and it is given that $x + y + z = 3$. We can write any point on this line as $(2\lambda - 2, -\lambda - 1, 3\lambda)$. which satisfies the plane

$$(2\lambda - 2) + (-\lambda - 1) + (3\lambda) = 3$$

$$\Rightarrow 4\lambda - 3 = 3$$

$$\Rightarrow \lambda = \frac{3}{2}$$

Thus, the point of intersection of plane is given by

$$\left(1, \frac{-5}{2}, \frac{9}{2}\right) \equiv C.$$

The point on the line is (–2, –1, 0) and direction ratio of *AB* (see the following figure) is

$$\frac{\alpha+2}{1} = \frac{\beta+1}{1} = \frac{\gamma}{1} = k.$$

Any general point on line *AB* is (k – 2, k – 1, k), which satisfies the equation. Therefore,

$$(k-2) + (k-1) + k = 3.$$
$$\Rightarrow k = 2$$

Therefore,

$$(\alpha, \beta, \gamma) \equiv (0, 1, 2)$$

Thus, the equation of line passing through *BC* is

$$\frac{x}{1} = \frac{y-1}{-7/2} = \frac{z-2}{5/2}$$
$$\Rightarrow \frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$$

6. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Then the probability that the problem is solved correctly by at least one of them is

(A) $\frac{235}{256}$ (B) $\frac{21}{256}$

(C) $\frac{3}{256}$ (D) $\frac{253}{256}$

Solution

Let us consider that

$$P(A) = \frac{1}{2};$$
$$P(B) = \frac{3}{4};$$
$$P(C) = \frac{1}{4};$$
$$P(D) = \frac{1}{8}.$$

Therefore,

$$\begin{aligned} P(A \cup B \cup C \cup D) &= 1 - \overline{P(A \cup B \cup C \cup D)} \\ &= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}) \\ &= 1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C})\, P(\bar{D}) \\ &= 1 - \frac{1}{2} . \frac{1}{4} . \frac{3}{4} . \frac{7}{8} \\ &= 1 - \frac{21}{256} \\ &= \frac{235}{256} \end{aligned}$$

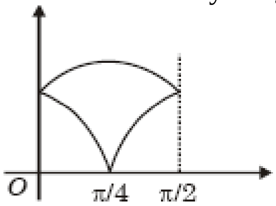
7. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

(A) $4(\sqrt{2}-1)$ (B) $2\sqrt{2}(\sqrt{2}-1)$

(C) $2(\sqrt{2}+1)$ (D) $2\sqrt{2}(\sqrt{2}+1)$

Solution

From the following figure that depicts the area enclosed by the given curves, we have



$$\int_0^{\pi/2} (\sin x + \cos x)dx - \left[\int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx\right]$$

$$= -|\cos x|_0^{\pi/2} + |\sin x|_0^{\pi/2} - \left[|\sin x|_0^{\pi/4} + |\cos x|_0^{\pi/4} - |\cos x|_{\pi/4}^{\pi/2} - |\sin x|_{\pi/4}^{\pi/2}\right]$$

$$= -(0-1) + (1-0) - \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 - \left(0 - \frac{1}{\sqrt{2}}\right) - \left(1 - \frac{1}{\sqrt{2}}\right)\right]$$

$$= 2 - \left[\sqrt{2} - 1 + \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}}\right]$$

$$= 2 - \left[2\sqrt{2} - 2\right]$$

$$= 4 - 2\sqrt{2}$$

$$= 2\sqrt{2}(\sqrt{2}-1)$$

8. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0.$ Then the equation of the curve is

(A) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (B) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

Solution

We have

$$\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$$

Substituting $y = vx,$ we get

$$v + x\frac{dv}{dx} = v + \sec v$$

$$\Rightarrow \cos v\, dv = \frac{dx}{x}$$

$$\Rightarrow \sin v = \ln\, x + c,$$

and as this passes through $\left(1, \frac{\pi}{6}\right)$, we have

$$\frac{1}{2} = c$$

and hence

$$\sin\left(\frac{y}{x}\right) = \ln x + \frac{1}{2}$$

9. Let $f:\left[\frac{1}{2},1\right] \to \mathbb{R}$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

(A) (2e – 1, 2e) (B) (e – 1, 2e – 1)

(C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

Solution

We have

$$\frac{dy}{dx} < 2y$$

That is,

$$e^{-2x}\frac{dy}{dx} < 2ye^{-2x}$$

$$\frac{d}{dx}(ye^{-2x}) < 0,$$

which implies that ye^{-2x} is a decreasing function. As $\frac{1}{2} < x < 1$, we have

$$e^{-1} > ye^{-2x} > y(1)e^{-2}$$

$$\Rightarrow e^{2x-1} > y > y(1)e^{2x-2}$$

$$\Rightarrow \int_{1/2}^{1} e^{2x-1}dx > \int_{1/2}^{1} ydx > \int_{1/2}^{1} y(1)e^{2x-2} > 0$$

Therefore,

$$0 < \int_{1/2}^{1} ydx < \frac{e-1}{2}$$

10. Let $f:\left[\frac{1}{2},1\right] \to \mathbb{R}$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

(A) (2e – 1, 2e) (B) (e – 1, 2e – 1)

(C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

Solution

Let us consider that

$$f(x) = x^2 - x\sin x - \cos x$$

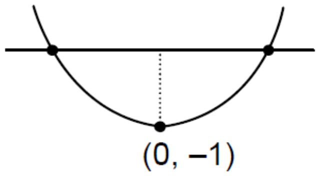
Therefore,

$$f'(x) = 2x \cdot x\cos x - \sin x + \sin x$$
$$= x(2 - \cos x)$$

$f(x)$ is increasing when $x > 0$; $f(x)$ is decreasing when $x < 0$. Therefore,

$$f(0) = -1$$
$$f(\infty) = \infty$$
$$f(-\infty) = \infty$$

Therefore, as shown in the following graph, the number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is two.



One or More than One Options Correct Type

This section contains **5 multiple choice questions**. Each question has four choices (A), (B), (C) and (D) out of which **ONE OR MORE** are correct.

11. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are

(A) 24 (B) 32
(C) 45 (D) 60

Solution

We have

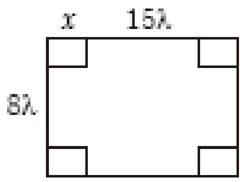
$$V = (8\lambda - 2x)(15\lambda - 2x)x$$
$$= 4x^3 - 46\lambda x^2 + 120\lambda^2 x$$

Differentiating with respect to x, we get

$$\frac{dV}{dx} = 12x^2 - 92\lambda x + 120\lambda^2 = 0 \qquad \text{at } x = 5$$

$$\Rightarrow \quad 60\lambda^2 - 230\lambda + 150 = 0$$
$$6\lambda^2 - 23\lambda + 15 = 0$$
$$(6\lambda - 5)(\lambda - 3) = 0$$

For $\lambda = 3$, the lengths of sides (as shown in the following figure) are obtained as 45, 24.



12. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s)

(A) 1056 (B) 1088
(C) 1120 (D) 1332

Solution

$$S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$$

Therefore,

$$S_n = -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + ... - (4n-3)^2 - (4n-2)^2 + (4n-1)^2 + (4n)^2$$
$$S_n = (3^2 - 1^2) + (4^2 - 2^2) + (7^2 - 5^2) + (8^2 - 6^2) + (11^2 - 9^2) + 12^2 - 10^2) + ... + (4n-1)^2 - (4n-3)^2 + (4n)^2 - (4n-2)^2$$
$$S_n = 2(1+3) + 2(4+2) + 2(7+5) + 2(8+6) + ... + 2(4n-1+4n-3) + 2(4n+4n-2)$$
$$S_n = 2[1+2+3+.....+4n] = \frac{2.4n(4n+1)}{2}$$

- From the value given in option (A), we get

$$4n(4n+1) = 1056$$
$$4n^2 + n = 264$$
$$4n^2 + n - 264 = 0$$
$$n = 8$$

- From the value given in option (B), we get

$$4n(4n + 1) = 1088,$$

which is not possible.

- From the value given in option (C), we get

$$4n(4n + 1) = 1120,$$

which is also not possible.

- From the value given in option (D), we get

$$4n(4n + 1) = 1332$$

for $n = 9$.

13. A line l passing through the origin is perpendicular to the lines

$$l_1 : (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}, \quad -\infty < t < \infty$$
$$l_2 : (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}, \quad -\infty < s < \infty$$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is (are)

(A) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (B) $(-1, -1, 0)$

(C) (1, 1, 1) (D) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

Solution

The equation of the line, l, is

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \lambda$$

The equation of the line, l_1 is

$$\frac{y+1}{2} = \frac{z-4}{2} = t$$

The equation of the line, l_2, is

$$\frac{x-3}{2}=\frac{y-3}{2}=\frac{z-2}{1}=s$$

The direction ratio of the line, l, is given by

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -2\hat{i}+3\hat{j}-2\hat{k}$$

The equation of the line, l, is

$$\frac{x}{-2}=\frac{y}{3}=\frac{z}{-2}=\lambda$$

The point of intersection of l and l_1 are as follows:

$$-2\lambda = 3+t \quad (1)$$
$$3\lambda = 2t-1 \quad (2)$$

Substituting the value of t, we get

$$3\lambda = 2(-2\lambda-3)-1$$

That is,

$$3\lambda = -4\lambda-6-1$$
$$7\lambda = -7$$
$$\lambda = -1$$

The point of intersection is $(2,-3,2)$. Therefore,

$$\sqrt{(3+2s-2)^2+(3+2s+3)^2+(2+s-2)^2}=\sqrt{17}$$
$$\Rightarrow 4s^2+4s+1+36+24s+4s^2+s^2=17$$
$$\Rightarrow 9s^2+28s+20=0$$

Therefore,

$$s=-2,\ -\frac{10}{9}$$

Thus, the intersection points are obtained as

$$(-1,-1,0) \text{ and } \left(\frac{7}{9},\frac{7}{9},\frac{8}{9}\right).$$

14. Let $f(x)=x\sin\pi x,\ x>0.$ Then for all natural numbers n, $f'(x)$ vanishes at

(A) A unique point in the interval $\left(n, n+\frac{1}{2}\right)$

(B) A unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

(C) A unique point in the interval $(n, n+1)$

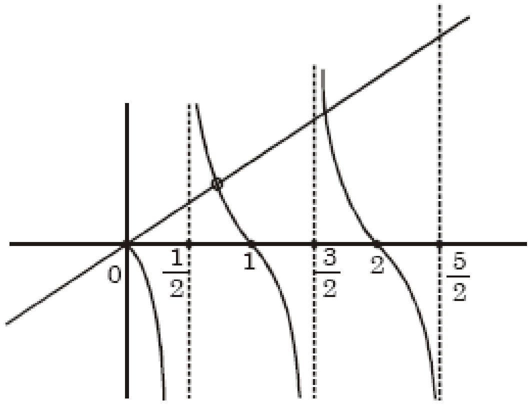
(D) Two points in the interval $(n, n+1)$

Solution

We have

$$f(x)=x\sin\pi x$$
$$f'(x)=\sin\pi x+\pi x\cos\pi x=0$$
$$\Rightarrow \quad -\tan\pi x=\pi x$$

It is clear from the following graph that $f'(x)$ has one root in $\left(n+\frac{1}{2}, n+1\right)$ and $f'(x)$ also has one root in $(n, n+1)$.



15. For 3 × 3 matrices M and N, which of the following statement(s) is (are) NOT correct?

(A) N^TMN is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(B) $MN – NM$ is skew symmetric for all symmetric matrices M and N

(C) MN is symmetric for all symmetric matrices M and N

(D) (adj M)·(adj N) = adj(MN) for all invertible matrices M and N

Solution

We have

$$(N^TMN)^T = -N^TM^T(N^T)^T = N^TM^TN$$

(A) If M is skew symmetric, then $(N^TMN)^T = -N^TMN$; therefore, it is concluded that it is skew symmetric.

If M is symmetric, then $(M^TMN)^T = N^TMN$; therefore, it is concluded that it is symmetric.

Hence, option (A) is correct.

(B) We have

$$\begin{aligned}(MN - NM)^T &= (MN)^T - (NM)^T \\ &= N^TM^T - M^TN^T \\ &= -(M^TM^T - N^TM^T) \\ &= -(MN - NM)\end{aligned}$$

Therefore, it is concluded that it is skew symmetric and hence option (B) is correct.

(C) $(MN)^T = N^TM^T$. Symmetricity and skew symmetricity depend on the nature of M and N, therefore, option (C) is incorrect.

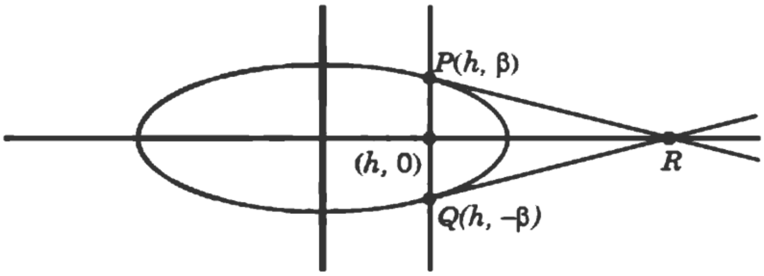
(D) adj(MM) = adj(N) adj M, therefore, option (D) is incorrect.

Integer Answer Type

This section contains **FIVE** questions. The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive.

16. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2\le h\le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2\le h\le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ _____.

Solution



We have

$$S \equiv \frac{x^2}{4} + \frac{y^2}{3} = 1$$

Let P and Q be (h, β) and $(h, -\beta)$, respectively (see the following figure). Therefore, R is

$$\left(\frac{4}{h}, 0\right).$$

Therefore,

$$\Delta = \frac{1}{2} \times 2\beta \times \left(\frac{4}{h} - h\right)$$
$$= \sqrt{3}\sqrt{1 - \frac{h^2}{4}} \times \left(\frac{4}{h} - h\right)$$
$$= \frac{\sqrt{3}}{2} \frac{(4 - h^2)^{3/2}}{h}$$

That is, $\frac{d\Delta}{dh} < 0$, from which it is clear that Δ is decreasing. That is,

$$\Delta_1 = \Delta\left(\frac{1}{2}\right) = \frac{15\sqrt{45}}{8}$$
$$\Delta_2 = \Delta(1) = \frac{9}{2}$$

Therefore,

$$\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 9$$

17. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = _____.

Solution

Let us consider that the consecutive terms be t_{r+2}, t_{r+1} and t_r. Therefore,

$$\frac{t_{r+1}}{t_r} = \frac{10}{5}$$
$$\Rightarrow \frac{(n+5) - (r+1) + 1}{r+1} = 2$$
$$\Rightarrow n - 3r + 3 = 0 \qquad (1)$$

Also we have

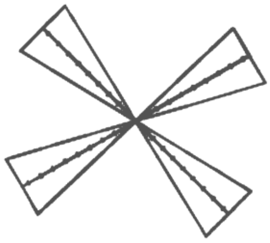
$$\frac{t_{r+2}}{t_{r+1}} = \frac{14}{10}$$
$$\Rightarrow \quad 5n - 12r + 6 = 0 \qquad (2)$$

Solving Eqs. (1) and (2), we get $n = 6$.

18. Consider the set of eight vectors $V = \{a\hat{i} + b\hat{j} + c\hat{k} : a, b, c \in \{-1, 1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is _____.

Solution

The eight vectors are as shown in the following figure.



The total number of vectors is given by

$$^8C_3 = 56$$

The total number of coplanar vectors is given by

$$2 \times (6 \times 2) = 24$$

That is,

$$56 - 24 = 32 \simeq 2^5$$

Hence, $p = 5$.

19. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1). Then

$$\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3} = \_\_\_\_\_.$$

Solution

We have

$$P(E_1)P(\bar{E}_2)P(\bar{E}_3) = \alpha \quad (1)$$

$$P(\bar{E}_1)P(E_2)P(\bar{E}_3) = \beta \quad (2)$$

$$P(\bar{E}_1)P(\bar{E}_2)P(E_3) = \gamma \quad (3)$$

$$P(\bar{E}_1)P(\bar{E}_2)P(\bar{E}_3) = \rho \quad (4)$$

Dividing Eq. (1) by Eq. (4), we get

$$\frac{P(E_1)}{P(\bar{E}_1)} = \frac{\alpha}{\rho} = \frac{\alpha}{\frac{\alpha\beta}{\alpha - 2\beta}} = \frac{\alpha - 2\beta}{\beta}$$

$$\Rightarrow \frac{P(\bar{E}_1)}{P(E_1)} = \frac{\beta}{\alpha - 2\beta}$$

$$\Rightarrow \frac{1 - P(E_1)}{P(E_1)} = \frac{\beta}{\alpha - 2\beta}$$

$$\Rightarrow \frac{1}{P(E_1)} - 1 = \frac{\beta}{\alpha - 2\beta}$$

$$\Rightarrow \frac{1}{P(E_1)} = \frac{\alpha - \beta}{\alpha - 2\beta}$$

$$\Rightarrow P(E_1) = \frac{\alpha - 2\beta}{\alpha - \beta} \quad (5)$$

Also

$$\frac{\alpha\beta}{\alpha - 2\beta} = \frac{2\beta\gamma}{\beta - 3\gamma}$$

$$\Rightarrow \quad \frac{\alpha}{\alpha-2\beta}=\frac{2\gamma}{\beta-3\gamma}$$

$$\Rightarrow \quad \alpha\beta-3\gamma\alpha=2\gamma\alpha-4\beta\gamma$$

$$\Rightarrow \quad \alpha\beta=5\gamma\alpha-4\beta\gamma$$

$$\gamma=\frac{\alpha\beta}{5\alpha-4\beta} \qquad (6)$$

Dividing Eq. (3) by Eq. (4), we get

$$\frac{P(E_3)}{P(\bar{E}_3)}=\frac{\gamma}{\rho}=\frac{\gamma}{\frac{\alpha\beta}{\alpha-2\beta}}=\frac{\gamma(\alpha-2\beta)}{\alpha\beta}$$

$$\frac{P(E_3)}{P(\bar{E}_3)}=\frac{\alpha-2\beta}{5\alpha-4\beta}$$

$$\frac{P(\bar{E}_3)}{P(E_3)}=\frac{5\alpha-4\beta}{\alpha-2\beta}$$

$$\frac{1-P(E_3)}{P(E_3)}=\frac{5\alpha-4\beta}{\alpha-2\beta}$$

$$\frac{1}{P(E_3)}=\frac{6\alpha-6\beta}{\alpha-2\beta}=\frac{6(\alpha-\beta)}{\alpha-2\beta}$$

$$P(E_3)=\frac{\alpha-2\beta}{6(\alpha-\beta)}$$

$$\frac{P(E_1)}{P(E_3)}=\frac{\frac{\alpha-2\beta}{\alpha-\beta}}{\frac{\alpha-2\beta}{6(\alpha-\beta)}}=6$$

Therefore,

$$\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3}=6.$$

20. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20$ = _____.

Solution

The smallest value of n for which

$$\frac{n(n+1)}{2}>1224$$

$$n(n+1)>2448$$

$$\Rightarrow \quad n>49$$

For $n=50$, we have

$$\frac{n(n+1)}{2} \quad \Rightarrow \quad 1275$$

Therefore,

$$k+(k+1)=1275-1224=51$$

Therefore, $k = 25$ and thus

$$k-20=5.$$