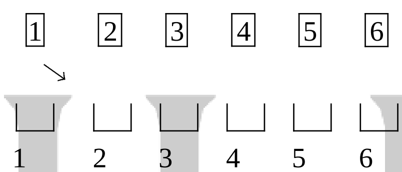
**JEE ADVANCED 2014**
**PAPER 2**
**MATHEMATICS**

## Only One Option Correct Type

This section contains **TEN** questions. Each has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four option(s) is correct.

**1.** Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 is always placed in envelope numbered 2. Then the number of ways it can be done is
(A) 264
(B) 265
(C) 53
(D) 67

**Solution**






This is problem of derangement.

Number of derangements of $n$ things is $=\lfloor n\sum_{k=0}^{n}\frac{(-1)^k}{\lfloor k}=\lfloor n\left[1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\cdots\frac{(-1)^n}{\lfloor n}\right]$

In the equation, two possibilities are there

I <u>When card two goes to envelope 1</u>

Derangement of 3, 4, 5, 6 cards is $\lfloor 4\left(\cancel{1}-\frac{1}{\cancel{\lfloor 1}}+\frac{1}{\cancel{\lfloor 2}}-\frac{1}{\cancel{\lfloor 3}}+\frac{1}{\cancel{\lfloor 4}}\right)$

$$=24\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}\right)=12-4+1$$

$$=\underline{9\text{ ways}}.$$

II. When card two does not go to envelope one.

Now it is derangement of 2, 3, 4, 5, 6 in envelopes 1 3 4 5 6 in $\lfloor 5\left(1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}-\frac{1}{\lfloor 5}\right)$

$$=120\left(\cancel{1}-\cancel{\frac{1}{1}}+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\right)=60-20+5-1$$

$$=44\text{ ways}$$

$\therefore$ Total number of way = 9 + 44 = 53

**2.** In a tringle the sum of two sides is $x$ and the product of the same two sides is $y$. If $x^2-c^2=y$, where $c$ is the third side of the triangle, then the ratio of the in radius to the circum-radius of the triangle is

(A) $\dfrac{3y}{2x(x+c)}$

(B) $\dfrac{3y}{2c(x+c)}$

(C) $\dfrac{3y}{4y(x+c)}$

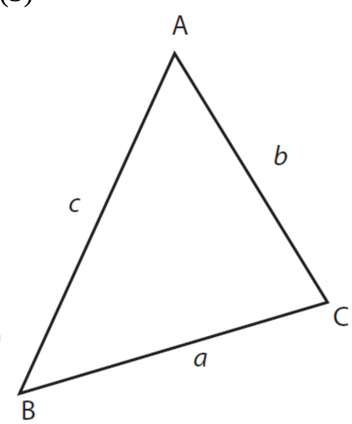
(D) $\dfrac{3y}{4c(x+c)}$

**Solution**

Let $a + b = x$ (1)

$ab = y$ (2)

$x^2 - c^2 = y$ (3)




From (1) $a + b + c = c + x$ (4)

From (2) $abc = cy$ (5)

Eqn. (4) $\Rightarrow$ $2s = c + x \Rightarrow 2\dfrac{\Delta}{r} = c + x \Rightarrow r = \dfrac{2\Delta}{c+x}$

$\because r = \dfrac{\Delta}{s}$

Eqn. (5) $\Rightarrow$ $4\Delta R = cy \Rightarrow R = \dfrac{cy}{4\Delta}$ $\quad \because R = \dfrac{abc}{4\Delta}$

$\therefore \dfrac{r}{R} = \dfrac{2\Delta}{c+x} \times \dfrac{4\Delta}{abc} = \dfrac{8\Delta^2}{(c+x)abc}$

$$= \frac{8 \times \left(\frac{1}{2}ab\sin C\right)^2}{(c+x)(abc)} = \frac{2a^{\cancel{2}}b^{\cancel{2}}\sin^2 C}{(c+x)\cancel{a}\,\cancel{b}c} \quad (6)$$

Now from Eqn. (3) $(a + b)^2 - c^2 = y$

or $2ab\cos c + 2ab = y$

or $2\cancel{ab}(1+\cos c) = \cancel{ab}$

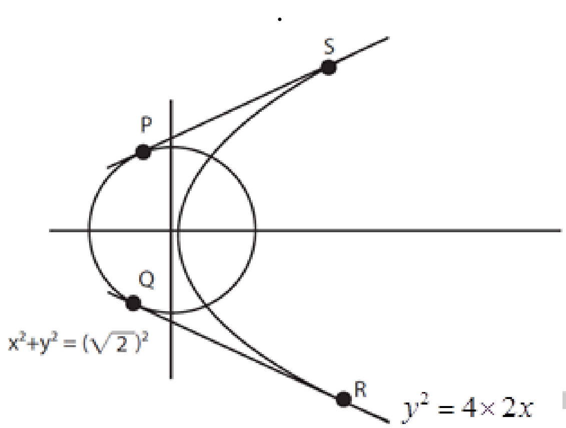
$\therefore \cos c = -\dfrac{1}{2}$

Now from Eqn. (6) $\dfrac{r}{R} = \dfrac{2ab(1-\cos^2 c)}{(c+x)c} = \dfrac{2ab\left(1-\frac{1}{4}\right)}{(c+x)c}$

$$= \frac{2ab \times \frac{3}{4}}{c(c+x)} = \frac{3ab}{2x(x+c)} = \frac{3y}{2c(x+c)}$$

**3.** The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points $P$, $Q$ and the parabola at the points $R$, $S$. Then the area quadrilateral $PQRS$ is
(A) 3
(B) 6
(C) 9
(D) 15

**Solution**




We know Eqn. of tangent to parabola with slope $m$ is $y = mx + \frac{2}{m}$ (1)

Since, Eqn. (1) is also a tangent to the circle, so length of perpendicular from the centre is equal to the radius.

$$\therefore \left| \frac{m(0) + \frac{2}{m} - 0}{\sqrt{m^2 + (-1)^2}} \right| = \sqrt{2}$$

$\therefore \frac{4}{m^2} = 2(m^2 + 1)$ or

$\Rightarrow 4 = 2m^2(m^2 + 1)$ or $2 = m^4 + m^2$

$\Rightarrow (m^2)^2 + m^2 - 2 = 0 \Rightarrow m^2 = \frac{-1 \pm \sqrt{1+8}}{2}$

$\therefore m^2 = \frac{-1 \pm 3}{2} = 1, -2 \because m^2 \neq -2 \therefore m^2 = 1$

$\Rightarrow m \pm 1$

$\therefore$ Tangents are $y = x + \frac{2}{1}$ and $y = -x - \frac{2}{1}$

Points of contact $\left(\frac{2}{1^2}, \frac{4}{1}\right)$ & $\left(\frac{2}{(-1)^2}, \frac{4}{-1}\right)$ (Parabola)

i.e. (2, 4) and (2, 4) )(S and R)

$\because \left(\frac{a}{m^2}, \frac{2a}{m}\right)$ for $\underline{y = mx + \frac{a}{m}}$

For point of contact of circle:
Solving $y = x + 2$ and $x^2 + y^2 = 2$
$\therefore\ x^2 + (x + 2)^2 \Rightarrow x^2 + x^2\ 4 + 4x - 2 = 0$
$\Rightarrow 2x^2 + 4x + 2 = 0 \Rightarrow x^2\ 2x + 1 = 0$
$\Rightarrow (x + 1)^2 = 0 \Rightarrow x = -1\ \therefore\ y = 1$
$\therefore$ (–1, 1) and (–1, –1), by symmetry are $P$ and $Q$.

Now area of the trapezium $PQRS$ is $\frac{1}{2}(PQ + RS) \times \text{Distance}$

i.e. $\frac{1}{2}\{(2+8)(1+2)\} = \frac{1}{\cancel{2}} \times \cancel{10}^{5} \times 3 = 15$

**4.** Three boys and two girls stand in a queue. The probability, that the number of boys ahead of every girl is at least one more then the number of girls ahead of her, is

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{2}{3}$

(D) $\frac{3}{4}$

**Solution**

According to question following possibilities are there

<u>Case 1:</u>
$G_1\ B_1\ G_2\ B_2\ B_3$
$B_1\ G_1\ B_2\ G_2\ B_3$
$G_1\ B_1\ B_2\ G_2\ B_3$
Girls separate
$\square B_1 \square B_2 \square B_3$

Out of 3 gaps, 2 are selected and girls are standing there in ${}^3C_2$ ways.

Next Boys and girls permute separately in $\lfloor 3 \times \lfloor 2$ ways.

$\therefore$ Number of ways $= {}^3C_2 \times \lfloor 3 \times \lfloor 2$
$= 3 \times 6 \times 2 = 36$

<u>Case 2:</u>
$G_1\ G_2\ B_1 B_2\ B_3$
$B_1\ G_1\ G_2 B_2\ B_3$
$B_1\ B_2\ G_1 G_2 B_3$ Not possible
Girls together

$\therefore$ Places selected in ${}^2C_1 = 2$ ways (Gaps) and then permutation is $\lfloor 3 \times \lfloor 2$ ways.

$\therefore$ Number ${}^2C_1 \times \lfloor 3 \times \lfloor 2 = 2 \times 6 \times 2 = 24$ ways.

$\therefore$ Probability $= \frac{36+24}{\lfloor 5} = \frac{\cancel{60}}{\cancel{120}\ 2} = \frac{1}{2}$

**5.** The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has
(A) only purely imaginary roots

(B) all real roots
(C) two real and two purely imaginary roots
(D) neither real nor purely imaginary roots

**Solution**

Given equation is $p(x) = 0$
Let it be written as $x^2 + c = 0$
where $c > 0$ ($\because$ purely imaginary roots)
$\therefore p(p(x)) = (p(x))^2 + c = 0$
or $(x^2 + c)^2 + c = 0$ or $x^4 + c^2 + 2cx^2 + c = 0$
or $(x^2)^2 + 2cx^2 + c^2 + c = 0$

$$\therefore\ x^2 = \frac{-2x \pm \sqrt{4c^2 - 4(c^2 + c)}}{2}$$

$$= \frac{-2c \pm 2\sqrt{c^2 - c^2 - c}}{2} = -c \pm ic \quad \text{where } c > 0$$

$$\therefore\ x = \pm\sqrt{-c \pm ic}$$

Hence $p(p(x)) = 0$ has neither real nor purely imaginary roots.

**Note:**
Let $c = 1$
Let us find square root of –1 + 2

$$\text{Let } x^2 = -1 + 2' = \sqrt{2}\left(\cos\left(\pi - \frac{\pi}{4}\right) + 2'\sin\left(\pi - \frac{\pi}{4}\right)\right)$$

$$\therefore\ x = 2^{\frac{1}{4}}\left\{\cos\left(\frac{3\pi}{4} + 2k\pi\right) + 2'\sin\left(\frac{3\pi}{4} + 2k\pi\right)\right\}^{\frac{1}{2}}$$

$k = 0, 1$

$$= 2^{\frac{1}{4}}\left\{\cos\frac{3\pi}{4} + 2'\sin\frac{3\pi}{4}\right\},\quad 2^{\frac{1}{4}}\left\{\cos\left(\frac{11\pi}{4}\right) + 2'\sin\left(\frac{11\pi}{4}\right)\right\}$$

**6.** For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has
(A) Infinitely many solutions
(B) Three solutions
(C) One solution
(D) No solution

**Solution**

Given equation is $\sin x + 2\sin x - \sin 3x = 3$

$$\Rightarrow\ \sin x + 4\sin x\cos x - 3\sin x + 4\sin^3 x = 3$$

$$\Rightarrow\ \sin x[1 + 4\cos x - 3 + 4\sin^2 x] = 3$$

$$\Rightarrow\ 1 + 4\cos x - 3 + 4(1 - \cos^2 x) = 3\cos ecx$$

$$\Rightarrow\ 2 + 4\cos x - 4\cos^2 x = 3\cos ecx$$

$$\Rightarrow\ -4\left\{\cos^2 x - \cos x - \frac{2}{42}\right\} = 3\cos ecx$$

$$\Rightarrow\ -4\left\{\cos^2 x - \cos x + \frac{1}{4} - \frac{1}{4} - \frac{1}{2}\right\} = 3\cos ecx$$

$$\Rightarrow\ -4\left\{\left(\cos x - \frac{1}{2}\right)^2 - \frac{3}{4}\right\} = 3\cos ecx$$

$$\text{min} = 0 \text{ at } x = \cos\frac{-11}{2}$$

$$\underbrace{\text{max} = \frac{3}{2} \text{ at } \pi}_{\text{min}}$$

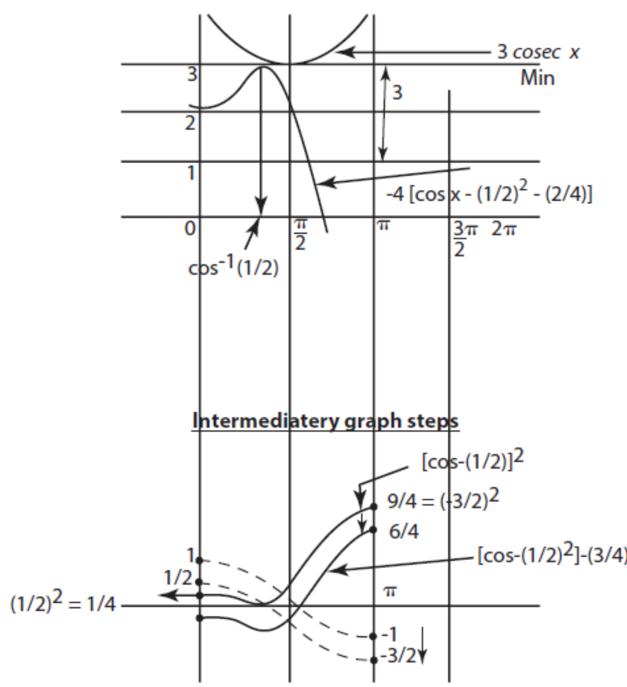
$$\text{min} = 3 \text{ at } \frac{\pi}{2}$$

$$\text{max} = (-4) \times \left(\frac{-3}{4}\right) = 3 \text{ at } x = \cos^{-1}\frac{1}{2} < \frac{\pi}{2}$$

$$\text{min} = (-\cancel{4}) \times \frac{6}{\cancel{4}} = -6 \text{ at } \pi$$

Through max of LHS = min of RHS but max of LHS occurs at a different point i.e. $\cos^{-1}\frac{1}{2} < \frac{\pi}{2}$ from min of RHS which occurs at $\frac{\pi}{2}$.

Within (0, $\pi$) rather in the whole domain, graph never meet.
Elaboration through graphs




**Note:** Eqn.(1) If max of LHS > $\propto$ and min of RHS < $\propto$ even then we cannot decided that there is a solution.

**7.** The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\operatorname{cosec} x)^{17}\, dx$ is equal to

(A) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16}\, du$

(B) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{17}\, du$

(C) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{17}\,du$

(D) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16}\,du$

**Solution**

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\text{cosec}x)^{17}\,dx$

Let $\text{cosec}x + \cot x = e^t$

$\therefore\ (-\text{cosec}x\cot x - \text{cosec}^2 x)dx = e^t dt$

$\Rightarrow\ -\text{cosec}x(\cot x + \text{cosec}x)dx = e^t dt$

$\Rightarrow\ -(\text{cosec}x)e^t dx = e^t dt$

$\therefore\ (\text{cosec}x)dx = -dt$

Now when $x = \dfrac{\pi}{4}$,

$\text{cosec}\dfrac{\pi}{4} + \cot\dfrac{\pi}{4} = e^t \qquad \therefore\ e^t = \sqrt{2} + 1$

$\therefore\ t = \log_e(1+\sqrt{2})$

when $x = \dfrac{\pi}{2}$, $\text{cosec}\dfrac{\pi}{2} + \cot\dfrac{\pi}{2} = e^t$

$\therefore\ 1 - 0 = e^t$

$\therefore\ t = 0$

$\therefore$ Also $\text{cosec}^2 x - \cot^2 x = 1$

$\Rightarrow\ \text{cosec}x - \cot x = \dfrac{1}{e^t} = e^{-t}$

$\therefore\ 2\ \text{cosec}\ x = e^t + e^{-t}$

$\therefore$ Integral reduces to

$$-\int_{\log_e(1+\sqrt{2})}^{0} (e^t + e^{-t})^{16}\, 2dt = +2\int_0^{\log_e(1+\sqrt{2})} (e^t + e^{-t})dt$$

**Note:** $t$ can be replaced by $\mu$ .

**8.** Coefficient of $x^{11}$ in the expansion of $(1 + x^2)^4\ (1 + x^3)^7\ (1 + x^4)^{12}$ is

(A) 1051

(B) 1106

(C) 1113

(D) 1120

**Solution**

Expanding by Binomial

Coeff. of $x^{11}$ in

$\{{}^4C_0(x^2)^0 + {}^4C_1(x^2)^1 + {}^4C_2(x^2)^2 + {}^4C_3(x^2)^3 + ({}^4C_4)(x^2)^4\}$

$\times\{{}^7C_0(x^3)^0 + {}^7C_1(x^3)^1 + ({}^7C_2)(x^3)^2 + ({}^7C_3)(x^3)^3 + ({}^7C_4)(x^3)^4 + \cdots\}$

further not required.

$\{{}^{12}C_0(x^4)^0 + {}^{12}C_1(x^4)^1 + {}^{12}C_2(x^4)^2 + ---\}$

Coeff. of $x^{11}$ in

$= \{1 + 4x^2 + 6x^4 + 4x^6 + x^8\}$

$\{1 + 7x^3 + 21x^6 + 35x^9\}$

$\{1 + 12x^4 + 64x^8\}$

Coeff. of $x^{11}$ in

$= \{1 + 7x^3 + 21x^6 + 35x^9 + 4x^2 + 28x^5 + 84r^8 + 140x^{11} + 6x^4 + 42x^7 + 126x^{10} + 4x^6 + 28x^9 + x^8 + 7x^{11}\} \times \{1 + 12x^4 + 66x^8\}$

$= \{1 + 7x^3 + 21x^6 + 35x^9 + 4x^2 + 28x^5 + 84r^8 + 140x^{11} + 6x^4 + 42x^7 + 126x^{10} + 4x^6 + 28x^9 + x^8 + 7x^{11}\} \times \{1 + 12x^4 + 66x^8\}$

$= 462 + 140 + 504 + 7$

$= 1113$

$\therefore$ Option (C) is correct.

**9.** Let $f : [0, 2] \rightarrow \mathbb{R}$ be a function which is continuous on [0, 2] and is differentiable on (0, 2) with $f$ (0) = 1.

Let $F(x)$ $F(x) = \int_0^{x^2} f(\sqrt{t})dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$ then $F(2)$ equals

(A) $e^2 - 1$
(B) $e^4 - 1$
(C) $e - 1$
(D) $e^4$

**Solution**

$\because \quad f(x) = \int_0^{x^2} f\left(\sqrt{d}\right)dt$

$\therefore \quad f'(x) = f\left(\sqrt{x^2}\right)\frac{d}{dx}(x^2) - f\left(\sqrt{0}\right)\frac{d}{dx}0$

(By Newton-Leibnitz rule)

$= 2xf\left(\sqrt{x^2}\right)$ (1)

Now according to question

$$f'(x) = f'(x) \Rightarrow 2xf\left(\sqrt{x^2}\right) = f'(x)$$

$\Rightarrow \; 2xf(x) = f'(x) \;\; \therefore \;\; x \in (0, 2)$

$\therefore \;\; \frac{f'(x)}{f(x)} = 2x \;\; \therefore \;\; \int \frac{f'(x)}{f(x)}dx = 2\int xdx$

$\Rightarrow \; \log_e f(x) = \frac{\cancel{2}x^2}{\cancel{2}} + c$

$\therefore \;\; f(x) = e^{x^2+c} = e^{x^2} \cdot e^c$

By initial condition $f(0) = 1$

$\therefore \; 1 = e^{o^2} \cdot e^c \;\Rightarrow e^c = 1$

$\therefore \;\; f(x) = e^{x^2} \;\; \therefore \;\; f\left(\sqrt{t}\right) = e^{\left(\sqrt{t}\right)^2} = e^t$

Now $f(x) = \int_0^{x^2} e^t dt = [e^t]_0^{x^2} = e^{x^2} - 1$

$\therefore \;\; f(2) = e^{2^2} - 1 = e^4 - 1$

**10.** The function $y = f(x)$ is the solution of the differential equation $\dfrac{dy}{dx}+\dfrac{xy}{x^2-1}=\dfrac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1.1)$ satisfying $f(0) = 0$. Then $\int_{\frac{\sqrt{3}}{2}} f(x)dx$ is

(A) $\dfrac{\pi}{3}-\dfrac{\sqrt{3}}{2}$

(B) $\dfrac{\pi}{3}-\dfrac{\sqrt{3}}{4}$

(C) $\dfrac{\pi}{6}-\dfrac{\sqrt{3}}{4}$

(D) $\dfrac{\pi}{6}-\dfrac{\sqrt{3}}{2}$

**Solution**

Differential Equation is $\dfrac{dy}{dx}+\dfrac{xy}{x^2-1}=\dfrac{x^4+2x}{\sqrt{1-x^2}}$

or $\dfrac{dy}{dx}-\left(\dfrac{x}{1-x^2}\right)y=\dfrac{x^4+2x}{\sqrt{1-x^2}}$ (1)

It is a linear differential eqn.

$\therefore$ I.F. $= e^{\int\left(\frac{-x}{1-x^2}\right)dx} = e^{\frac{1}{2}\log(1-x^2)}$

$= e\log_e\sqrt{1-x^2}$

Now multiplying I.F. with (1)

$\therefore\ \sqrt{1-x^2}\dfrac{dy}{dx}-\sqrt{1-x^2}\dfrac{x}{(1-x^2)}y=x^4+2x$

$\Rightarrow\ \dfrac{d}{dx}(\sqrt{1-x^2}\,y)=x^4+2x$

$\Rightarrow\ \int d(\sqrt{1-x^2}\,y)=\int(x^4+2x)dx$

$\Rightarrow\ \sqrt{1-x^2}\,y=\dfrac{x^5}{5}+\dfrac{2x^2}{2}+c$ (2)

Using initial conditions

$$\sqrt{1-x^2}(0)=0+0+c\ \ \therefore\ c=0$$

$\therefore$ (2) given $\sqrt{1-x^2}\,y=\dfrac{x^5}{5}+x^2$

$\therefore\ y=\dfrac{\frac{x^5}{5}+x^2}{\sqrt{1-x^2}}=\dfrac{x^5}{5\sqrt{1-x^2}}+\dfrac{x^2}{\sqrt{1-x^2}}$

Now

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx=\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\frac{x^5}{5\sqrt{1-x^2}}dx+\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\frac{x^2}{\sqrt{1-x^2}}dx$$

$$=0+2\int_0^{\frac{\sqrt{3}}{2}}\frac{x^2}{\sqrt{1-x^2}}dx \qquad (3)$$

$\because$ even function

Now solving $\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$

Putting $x = \sin\theta$ $\quad\therefore\ dx = \cos\theta\, d\theta$

$\therefore\ \int_0^{\frac{\sqrt{3}}{2}} \frac{\sin^2\theta\cos\theta d\theta}{\cos\theta} = \int_0^{\frac{\pi}{3}} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$

$= \left[\frac{1}{2}\theta\right]_0^{\frac{\pi}{3}} - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\times\frac{\pi}{3} - \frac{1}{4}\left[\sin\frac{2\pi}{3}\right]$

$= \frac{\pi}{6} - \frac{1}{4}\frac{\sqrt{3}}{2} = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

$\therefore$ From Eqn. (3)

Answer is $2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$.

**Paragraph Type**

This section contains **THREE** paragraphs, each describing theory, experiments, data etc. Based on each paragraph, there will be **TWO** questions. Each question has four options (A), (B), (C) and (D). **ONLY ONE** of these four option(s) is correct.

*Paragraph for Questions 11 and 12:* Box 1 contains three cards bearing numbers 1, 2, 3, box 2 contains five five bearing numbers 1, 2, 3, 4, 5; and box 3 contains seven cards bearing numbers 1, 2, 3, 4, 5, 6, 7. A card is drawn from each of the boxes. Let $x_i$ be number of the card drawn from the $i^{th}$ box, $i$ = 1,2,3.

**11.** The probability that $x_1 + x_2 + x_3$ is odd, is

(A) $\frac{29}{105}$

(B) $\frac{53}{105}$

(C) $\frac{57}{105}$

(D) $\frac{1}{2}$

**Solution**

$\underset{1}{\underline{|(1)(2)(3)|}}\ \underset{2}{\underline{|(1)(2)(3)(4)(5)|}}\ \underset{3}{\underline{|(1)(2)(3)(4)(5)(6)(7)|}}$

For $x_1 + x_2 + x_3$ to be odd, either all the numbers are odd.

2 × 3 × 3 × 4 = 24 ways

or one odd and two even in

$\underbrace{2\times2\times3}_{\text{odd from Ist}} + \underbrace{1\times3\times3}_{\text{odd from second}} + \underbrace{1\times2\times4}_{\text{odd from 3rd}}$

= 12 + 9 + 8 = 29 ways

$\therefore$ Total number of ways = 24 + 29 = 53 ways.

All possibilities = 3 × 5 × 7 = 105

$\therefore$ Probability $= \frac{53}{105}$

**12.** The probability that $x_1$, $x_2$, $x_3$ are in an arithmetic progression, is

(A) $\frac{9}{105}$

(B) $\frac{10}{105}$

(C) $\frac{11}{105}$

(D) $\frac{7}{105}$

**Solution**

For $x_1$, $x_2$, $x_3$ to be in A.P.

$2x_2 = x_1 + x_3$

$\therefore$ we require $(x_1 + x_3)$ to be even.

$\therefore$ either both even in $= 1 \times 3 = 3$ ways

or both $x_1$, $x_3$ odd is $2 \times 4 = 8$ ways

All possibilities $= 3 \times 5 \times 7 = 105$

$\therefore$ Probability $= \frac{\text{Favorable ways}}{\text{All possibilities}} = \frac{8+3}{105} = \frac{11}{105}$

***Paragraph for Questions 13 and 14:*** Let $a$, $r$, $s$, $t$ be nonzero real numbers, Let $P(at^2, 2at)$, $Q$, $R\,(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that $PQ$ is the focal chord and lines $QR$ and $PK$ are parallel, where $K$ is the point $(2a, 0)$.

**13.** The value $r$ is

(A) $-\frac{1}{t}$

(B) $\frac{t^2+1}{t}$

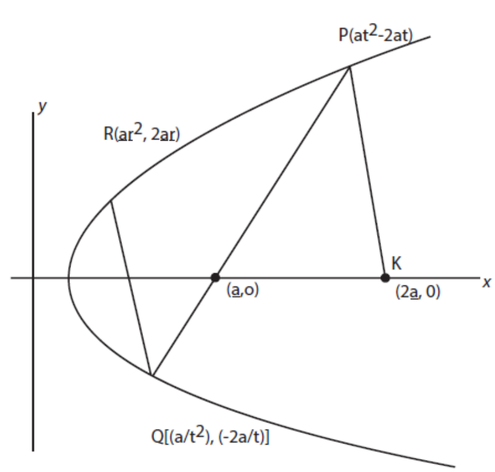
(C) $\frac{1}{t}$

(D) $\frac{t^2-1}{t}$

**Solution**

Since $QR \parallel PK$

$\therefore$ slope of $QR$ = Slope of $PK$




$$\Rightarrow \qquad \frac{2ar + \frac{2a}{t}}{2r^2 - \frac{a}{t^2}} = \frac{2at}{at^2 - 2a}$$

$$\Rightarrow \frac{r + \frac{1}{t}}{r^2 - \frac{1}{t^2}} = \frac{t}{t^2 - 2}$$

$$\Rightarrow \frac{\left(r + \frac{1}{t}\right)}{\left(r + \frac{1}{t}\right)\left(r - \frac{1}{t}\right)} = \frac{t}{t^2 - 2}$$

$\Rightarrow tr - 1 = t^2 - 2$

$\therefore \; r = \frac{t^2 - 1}{t}$

$\because \; PQ$ is a focal chord

$\therefore \; t_1 t = -1 \therefore t_1 = -\frac{1}{t}$

$\therefore \; (Q(at_1^2, 2at_1))$

$= Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

**14.** If $st = 1$. then the tangent at $P$ and the normal at $S$ to the parabola meet at a point whose ordinate is

(A) $\frac{(t^2 + 1)^2}{2t^3}$

(B) $\frac{a(t^2 + 1)^2}{2t^3}$

(C) $\frac{a(t^2 + 1)^2}{t^3}$

(D) $\frac{a(t^2 + 2)^2}{t^3}$

**Solution**

$\because\ st = 1,\ \therefore\ s = \frac{1}{t},\ \therefore\ S\left(a\left(\frac{1}{t}\right)^2, \frac{2a}{t}\right) = S\left(\frac{a^2}{t^2}, \frac{2a}{t}\right)$

Now Eqn. of tangent of $P$ is

$y^t = x + at^2$ (1)

Eqn. of normal at $S$ is

$y = -\Delta x + 2as + as^3$

$= -\frac{x}{t} + \frac{2a}{t} + \frac{a}{t^3}$ (2)

From Eqn. (1) $x = \underline{yt - at^2}$

From Eqn. (2) $x = \left(\frac{2a}{t} + \frac{a}{t^3} - y\right)t$

$\therefore\ yt - at^2 = 2a + \frac{a}{t^2} - yt$

$\therefore\ 2yt = at^2 + \frac{a}{t^2} + 2a = a\left(t^2 + \frac{1}{t^2} + 2\right)$

$= a\left(t + \frac{1}{t}\right)^2$

$\therefore\ y = \frac{a\left(t - \frac{1}{t}\right)^2}{2t} = \frac{a(t^2+1)}{2t^3}$

*Paragraph for Questions 15 and 16:* Given that for each $a \in (0, 1)$.

$$\lim_{h\to 0} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$$

exists. Let this limit be $g(a)$. In addtion, it is given that function $g(a)$ is differentiable on (0, 1).

**15.** The value of $g\left(\frac{1}{2}\right)$ is

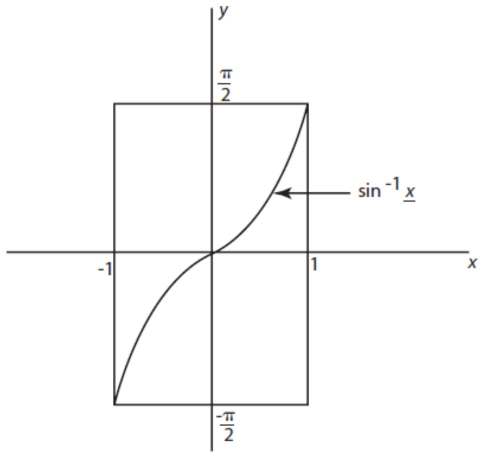
(A) $\pi$

(B) $2\pi$

(C) $\frac{\pi}{2}$

(D) $\frac{\pi}{4}$

**Solution**




Given $g(a)=\lim_{h\to 0+}\int_h^{1-h} t^{-a}(1-t)^{a-1}dt$

$$g\left(\frac{1}{2}\right)=\lim_{h\to 0+}\int_h^{1-h} t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt=\int_{\substack{h\\h\to 0}}^{1-h}\frac{1}{\sqrt{t(1-t)}}dt$$

$$={}_{h\to 0+}\int_h^{1-h}\frac{1}{\sqrt{t-t^2}}dt={}_{h\to 0+}\int_h^{1-h}\frac{1}{\sqrt{-\left(t^2-t+\frac{1}{4}-\frac{1}{4}\right)}}dt$$

$$={}_{h\to 0+}\int_h^{1-h}\frac{1}{-\left\{t-\frac{1}{2}\right\}^2+\frac{1}{4}}dt={}_{h\to 0+}\int_h^{1-h}\frac{1}{\left(\frac{1}{2}\right)^2-\left(t-\frac{1}{2}\right)^2}dt$$

$$={}_{h\to 0+}\left[\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_h^{1-h}={}_{h\to 0+}\left[\sin^{-1}(2t-1)\right]_h^{1-h}$$

$$={}_{h\to 0+}[\sin^{-1}\{2-2h-1\}-\sin^{-1}(2h-1)]$$

$$={}_{h\to 0+}\sin^{-1}(1-2h)-{}_{h\to 0+}\sin^{-1}(2h-1)$$

$$=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)$$

$$=\pi$$

**16.** The value of $g'\left(\frac{1}{2}\right)$ is

(A) $\frac{\pi}{2}$

(B) π

(C) $-\frac{\pi}{2}$

(D) 0

**Solution**

$$g(a)=\lim_{h\to 0+}\int_h^{1-h} t^{-a}(1-t)^{a-1}dt \qquad (1)$$

Now $g(1-a)=\lim_{h\to 0+}\int_h^{1-h} t^{-(1-a)}(1-t)^{1-a-1}dt$

$$= \lim_{h\to 0+} \int_h^{1-h} t^{a-1}(1-t)^{-a}\, dt$$

$$= \lim_{h\to 0+} \int_h^{1-h} (\not{h}+1-\not{h}-t)^{a-1}\{1-(\not{h}+1-\not{h}-t)\}^{-a}\, dt$$

$$= _{h\to 0+}\int_h^{1-h} (1-t)^{a-1}(\not{1}-\not{1}+t)^{-a}\, dt$$

$$= _{h\to 0+}\int_h^{1-h} t^{-a}(1-t)^{a-1}\, dt \qquad (2)$$

From Eqn. (1) & (2) $g(a) = g(1-a)$

$\therefore g'(a) = -g'(1-a)$

when $a = \frac{1}{2}$, $g'\left(\frac{1}{2}\right) = -g'\left(\frac{1}{2}\right)$

$\Rightarrow g'\left(\frac{1}{2}\right) = 0$

**Matching List Type**

This section contains **FOUR** questions, each having two matching lists. Choice for the correct combination of elements from List-I and List-II are given as options (A), (B), (C) and (D), out of which **ONLY ONE** is correct

**17.**

| | **List-I** | | **List-II** |
|---|---|---|---|
| **P.** | The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)dx = 1$, is | **1.** | 8 |
| **Q.** | The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is | **2.** | 2 |
| **R.** | $\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals | **3.** | 4 |
| **S.** | $\dfrac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_0^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals | **4.** | 0 |

| | **P** | **Q** | **R** | **S** |
|---|---|---|---|---|
| (A) | 3 | 2 | 4 | 1 |
| (B) | 2 | 3 | 4 | 1 |
| (C) | 3 | 2 | 1 | 4 |
| (D) | 2 | 3 | 1 | 4 |

**Solution**

For (P) in List I:

Let $f(x) = ax^2 + bx$, $\because f(0) = 0$

Also $\int_0^1 fxdx = 1 \Rightarrow \int_0^1 (ax^2 + bx)dx = 1$

$$\Rightarrow \left[\frac{ax^3}{3}+\frac{bx^2}{2}\right]_0^1 = 1 \Rightarrow \frac{a}{3}+\frac{b}{2}=1$$

$\because a \geq 0, b \geq 0$, $\therefore$ $\frac{0}{3}+\frac{2}{2}=1$, $\therefore$ $a = 3$, $b = 0$

$$\Rightarrow \frac{3}{3}+\frac{0}{2}=1 \quad \because a=3, b=0$$

$\therefore$ Possible polynomials are $f(x) = bx$ or $f(x) = 3x^2$

$\therefore$ (P) $\rightarrow$ (2)

$$f(x)=\sin x^2+\cos x^2=\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x^2+\frac{1}{\sqrt{2}}\cos x^2\right]=\sqrt{2}\sin\left(x^2+\frac{\pi}{4}\right)$$

Now $f(x)$ is maximum when $x^2+\frac{\pi}{4}=\frac{\pi}{2}$

Generalizing for whole domain:

$x^2+\frac{\pi}{4}=\frac{\pi}{2}+2k\pi$, where $k$ is integer and $2\pi$ is period

$\therefore$ $x^2=2k\pi+\frac{\pi}{4}$, $\because$ $\pi \approx 3.14$

when $k = 0$, $x^2=\frac{\pi}{4}$ $\quad \in [0, 13]$

when $k = 1$, $x^2=\frac{9\pi}{4}$ $\quad \in [0, 13]$

For (S) in List I:

Let $f(x)=\cos 2x fx\left(\frac{1+x}{1-x}\right)$

$\therefore$ $f(-x)=\cos 2x fx\left(\frac{1-x}{1+x}\right)$

$$\Rightarrow f(x)+f(-x)=\cos 2x\left[f_n\left(\frac{\cancel{1+x}}{\cancel{1-x}}\times\frac{\cancel{1-x}}{\cancel{1+x}}\right)\right]$$

$$=0$$

$\therefore$ $f(x)$ is odd function

$$\therefore \int_{-\frac{1}{2}}^{\frac{1}{2}}\cos 2x\, fx\left(\frac{1+x}{1-x}\right)dx=0$$

$\therefore$ $(S) \rightarrow (4)$

$\therefore$ Denominator will be non-zero.

**18.**

| | **List-I** | | **List-II** |
|---|---|---|---|
| **P.** | Let $y(x) = \cos(3\cos^{-1} x)$, $x \in [-1, 1]$, $x \neq \pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ equals | **1.** | 1 |
| **Q.** | Let $A_1, A_T \ldots A_n$ $(n > 2)$ be the vertices of a regular polygon of $n$ sides with its centre at the origin. | **2.** | 2 |

Let $\overrightarrow{a_k}$ be the position vector of the point $A_k$, $k - 1.2.\ldots.n$.

If $\left|\sum_{k=1}^{n-1}(\overrightarrow{a_k}\times\overrightarrow{a_{k-1}})\right| = \left|\sum_{k=1}^{n-1}(\overrightarrow{a_k}\cdot\overrightarrow{a_{k+1}})\right|$, then the minimum value of $n$ is

**R.** If the normal from the point $P(h, 1)$ on the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ is perpendicular to the line $x + y = 8$. then the value of $h$ is — **3.** 8

**S.** Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ is — **4.** 9

| | P | Q | R | S |
|---|---|---|---|---|
| (A) | 1 | 3 | 2 | 1 |
| (B) | 2 | 4 | 3 | 1 |
| (C) | 4 | 3 | 1 | 2 |
| (D) | 2 | 4 | 1 | 3 |

**Solution**

For (P) in List I:

$y(x) = \cos(3\cos^{-1}x)$

$$\therefore \frac{dy}{dx} = -\{\sin(2\cos^{-1}x)\}3\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{3\sin(3\cos^{-1}x)}{\sqrt{1-x^2}}$$

$$= \left\{\sqrt{1-x^2}\frac{dy}{dx}\right\}^2 = \left\{2\sin(3\cos^{-1}x)\right\}^2$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = 9\sin^2(3\cos^{-1}x)$$

$$= 9\{1-\cos^2(3\cos^{-1}x)\}$$

$$= 9\{1-y^2\}$$

None differentiating

$$(1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2(-2x) = 9\left\{-2y\frac{dy}{dx}\right\}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -9y$$

$$\Rightarrow (x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 9y \quad \therefore P \to 4.$$

When $k = 2$, $x^2 = \frac{17\pi}{4} \notin [0,13]$

$\therefore$ There are only two values of $x^2$ and hence 4 values of $x$.

$\therefore$ (Q) $\to$ (3)

For (R) in List I:

$$\int_{-2}^{2}\frac{3x^2}{1+e^x}dx = \int_{-2}^{0}\frac{3x^2}{1+e^x}dx + \int_{0}^{2}\frac{3x^2}{1+e^x}dx \qquad (1)$$

Now $\int_{-2}^{0} \frac{3x^2}{1+e^x} dx = -\int_{2}^{0} \frac{3(-t)^2}{1+e^{-t}} dt$

Putting $x = t$

$\therefore\ x = -2 \Rightarrow t = 2$

$x = 0 \Rightarrow t = 0$

$$= -\int_{2}^{0} \frac{3t^2}{1+e^{-t}} dt$$

$$= +\int_{0}^{2} \frac{3x^2}{1+e^{-x}} dx$$

Now from (1)

$$\int_{-2}^{2} \frac{3x^2}{1+e^x} dx = \int_{0}^{2} \frac{3x^2}{1+e^{-x}} dx + \int_{0}^{2} \frac{3x^2}{1+e^x} dx$$

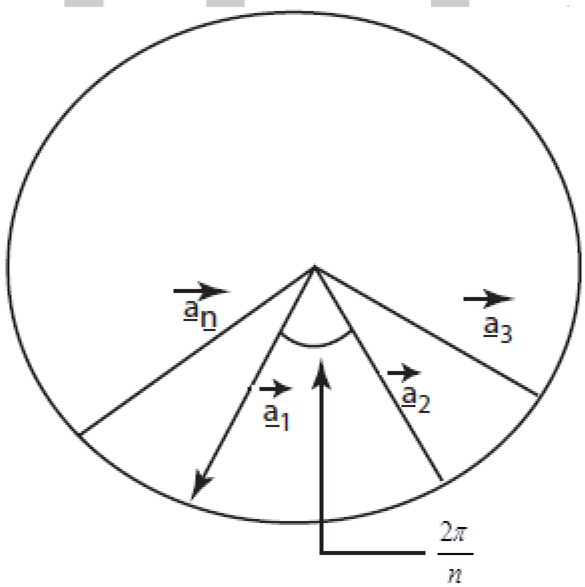
$$= \int_{0}^{2} 3x^2 \left( \frac{1}{1+\frac{1}{e^x}} + \frac{1}{1+e^x} \right) dx = \int_{0}^{2} 3x^2 \left( \frac{e^x}{1+e^x} + \frac{1}{1+e^x} \right)_{dx}$$

$$= \int_{0}^{2} (3x^2) \frac{e^x+1}{e^x+1} dx = 3\left[ \frac{x^3}{3} \right]_0^2$$

$$= 2^3 - 0 = 8 \quad \therefore R \to 1$$

For (Q) in List I:

$$\left| \sum_{k=1}^{n-1} (\vec{a}_k \times \vec{a}_{k\to 1}) \right| = | \vec{a}_1 \times \vec{a}_2 + \vec{a}_2 \times \vec{a}_3 + \cdots \vec{a}_{n-1} \times \vec{a}_n |$$




$$= (a^2 + a^2 + \cdots a^2) \sin \tfrac{2\pi}{n}$$

$$= (n-1)a^2 \sin \tfrac{2\pi}{n} \qquad (1)$$

$\because$ all $|\vec{a}_1|$ equal

Also $\left| \sum_{k=1}^{n-1} (\vec{a}_k \cdot \vec{a}_{k+1}) \right| = (a^2 + a^2 + \cdots a^2) \cos \frac{2\pi}{n}$

$$= (n-1)a^2 \cos\frac{2\pi}{n} \qquad (2)$$

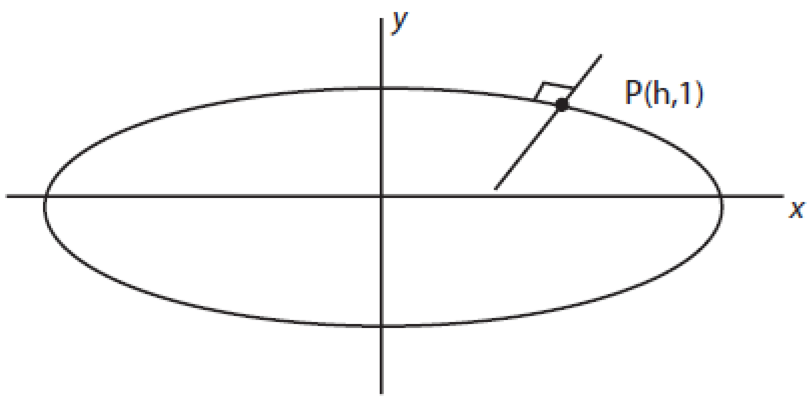
From (1) & (2) $\sin\frac{2\pi}{n} = \cos\frac{2\pi}{n}$

$\Rightarrow \tan\frac{2\pi}{n} = 1 \Rightarrow \frac{2\pi}{n} = \frac{\pi}{4} \quad \therefore n = 8$

$\therefore$ (Q) $\rightarrow$ (3)

For (R) in List I:

Eqn. of normal at $(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$ is




$\frac{\sqrt{6}x}{\cos\theta} - \frac{\sqrt{3}y}{\sin\theta} = 3 \qquad (1)$

$\because \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

$\therefore$ Slope of this normal is

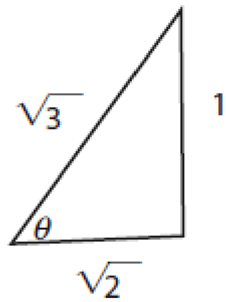
$$= \left(-\frac{\sqrt{6}}{\cos\theta}\right) \Big/ \left(\frac{-\sqrt{3}}{\sin\theta}\right) = \frac{\sqrt{6}}{\sqrt{3}} \frac{\sin\theta}{\cos\theta} = \sqrt{2}\tan\theta$$

$\because$ $(h, 1)$ lies an this normal

$\therefore \frac{\sqrt{6}h}{\cos\theta} - \frac{\sqrt{3}l}{\sin\theta} = 3 \qquad (2)$

Now this normal is $\perp$ to x + y = 8 $\therefore$ its slope in $\frac{-1}{-1} = 1$

$\therefore \sqrt{2}$ tan θ = 1 ⇒ tan θ = $\frac{1}{\sqrt{2}}$




$$\therefore\ \cos\theta = \frac{\sqrt{2}}{\sqrt{3}},\quad \sin\theta = \frac{1}{\sqrt{3}}$$

$$\therefore\ (2) \Rightarrow \frac{\sqrt{6}h}{\frac{\sqrt{2}}{\sqrt{3}}} - \frac{\sqrt{3}}{\frac{1}{\sqrt{3}}} = 3$$

$$\Rightarrow \sqrt{\frac{\overset{3}{\cancel{6}} \times 3}{\cancel{2}}}\quad h = 6$$

$$\therefore\ h = \frac{\cancel{6}^{2}}{\cancel{3}}\ \therefore\ h = 2$$

$\therefore$ (R) $\rightarrow$ (2)

For (S) in List I:

$$\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow\ \tan^{-1}\left\{\frac{\frac{1}{2x+1} + \frac{1}{4x+1}}{1 - \frac{1}{(2x+1)(4x+1)}}\right\} = \tan^{-1}\left(\frac{3}{x^2}\right)$$

$$\Rightarrow\ \tan^{-1}\left(\frac{\frac{4x+1+2x+1}{8x^2+6x+1}}{\frac{8x^2+6x+1-1}{8x^2+6x+1}}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow\ \tan^{-1}\left(\frac{6x+2}{8x^2+6x}\right) = \tan^{-1}\frac{2}{x^2}$$

$$\therefore\ \ 6x^3 + 2x^2 = 16x^2 + 12x$$

$$\Rightarrow\ 2x\left\{3x^2 + x - 8x - 6\right\} = 0$$

$$\Rightarrow\ x(3x^2 - 7x - 6) = 0$$

$$\Rightarrow\ x\left\{3x^2 - 9x + 25 - 6\right\} = 0$$

$$\Rightarrow\ \left\{3x(x-3) + 2(x-3)\right\} = 0$$

or $\ x(x-3)(3x+2) = 0$

$$\therefore\ x = 0,\ 3,\ -\frac{2}{3}$$

$\therefore$ Number of +ve solutions = 1

$\therefore$ (S) $\rightarrow$ (1)

**19.** Let $f_1$: $\mathbb{R} \to \mathbb{R}$ , $f_2$: $[0, \infty) \to \mathbb{R}$ , $f_3$: $\mathbb{R} \to \mathbb{R}$ and $f_4$: $\mathbb{R} \to [0, \infty)$ be defined by

$$f_1(x) = \begin{cases} |x| & \text{if} \quad x < 0. \\ e^x & \text{if} \quad x \geq 0: \end{cases}$$

$$f_2(x) = x^2$$

$$f_3(x) = \begin{cases} \sin x & \text{if} \quad x < 0. \\ x & \text{if} \quad x \geq 0 \end{cases}$$

and

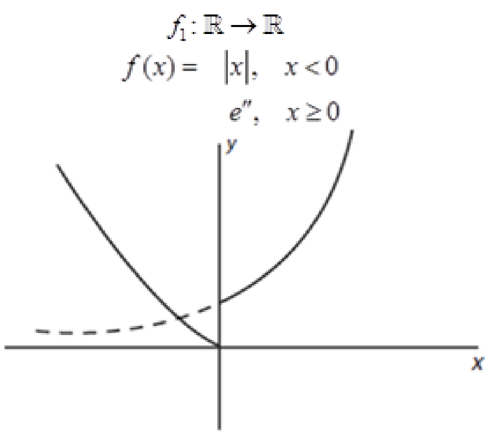
$$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if} \quad x < 0 \\ f_2(f_1(x)) - 1 & \text{if} \quad x \geq 0 \end{cases}$$

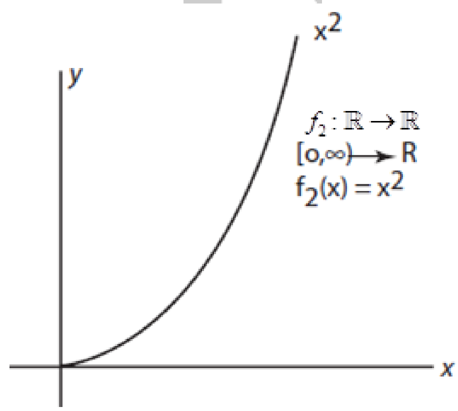
| | List-I | | List-II |
|---|---|---|---|
| **P.** | $f_4$ is | 1. | onto but not one-one |
| **Q.** | $f_3$ is | 2. | neither continuous nor one-one |
| **R.** | $f_2$ o$f_1$ is | 3. | differentiable but not one-one |
| **S.** | $f_2$ is | 4. | continuous and one-one |

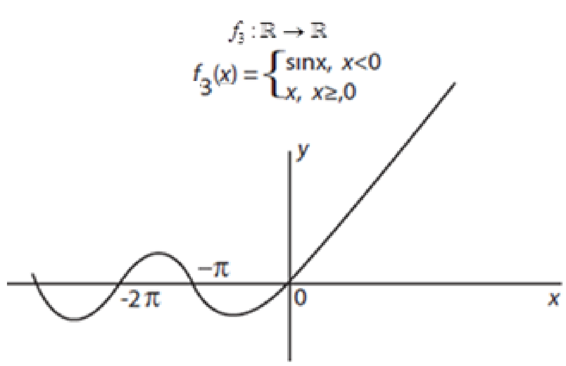
| | **P** | **Q** | **R** | **S** |
|---|---|---|---|---|
| (A) | 3 | 1 | 4 | 2 |
| (B) | 1 | 3 | 4 | 2 |
| (C) | 3 | 1 | 2 | 4 |
| (D) | 1 | 3 | 2 | 4 |



**Solution**



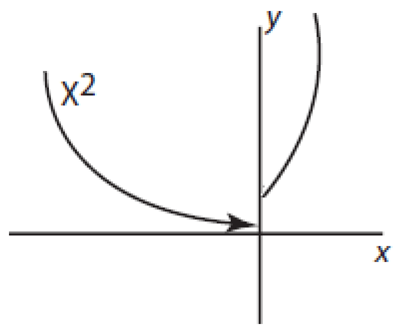



$$f_2(f_1(x) = (f_1(x))^2 = \left| \begin{array}{llr} \rightarrow & |x|^2, \pi < 0 \quad \& & |x| \text{ valid} \\ & & \uparrow \\ & & \text{for } f,(x) \in R \\ & & \downarrow \\ \rightarrow & (e^x)^2, \pi \geq 0 \quad \& & e^x \text{valid} \end{array} \right.$$

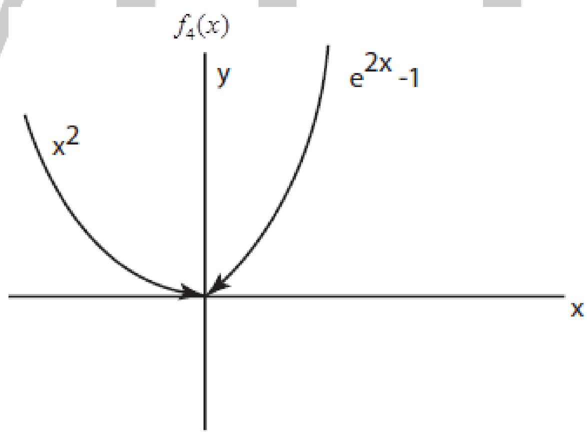
$$\text{Range } f_1 \subset \text{Dom } f_2$$

$$\therefore f_2(f_1(x)) = \quad x^2 \quad \pi < 0$$

$$= \quad e^{2x} \quad \pi \geq 0$$




$$\therefore f_4(x) = \begin{cases} x^2, x < 0 \\ e^{2x} - 1, \; x \geq 0 \end{cases}$$




For (P) in List I:

$f_4$ Dom. R

Range $[0 \; \infty]$

Codomain = $[0 \; \infty]$, $\therefore$ onto

Now LHL at 0 = $\lim_{x \to 0-} \quad x^2 = 0$

RHL at 0 = $\lim_{x \to 0+} e^{2x} - 1 = 1 - 1 = 0$

$f_4(0) = 0$

$\therefore$ Continuous at 0

Not LHD at $0 \underset{h \to 0}{=} \dfrac{f(0-h) - f(0)}{-h} \underset{h \to 0}{=} \dfrac{h^2 - 0}{-h} \underset{h \to 0}{=} -h = 0$

RHD at 0 $\underset{h\to 0}{=} \dfrac{f(0+h)-f(0)}{h} \underset{h\to 0}{=} \dfrac{e^{2h}-1-0}{h}$

$= \underset{h\to 0}{\dfrac{e^{2h}-1}{2h}}.2 = 1\cdot 2 = 2$

$\therefore$ Not derivable at 0.
It is not one-one (obvious from graph)
$\therefore (P) \to (1)$

For (Q) in List I:
$f_3$ in neither one-one nor onto.
It is derivability at 0.

LHD = $_{h\to 0}\dfrac{\sin(0-h)-0}{-h} =_{h\to 0} \dfrac{-\sinh}{-h} = \pm 1$

RHD = $_{h\to 0}\dfrac{0+h-0}{h} = 1$

$\therefore$ It is derivable at 0.
$\therefore (Q) \to (3)$
$(R) \to (2)$ (obvious from graph).
$(S) \to (4)$ (obvious from graph.)

**20.** Let $z_k = \cos\left(\dfrac{2k\pi}{10}\right) + i\sin\left(\dfrac{2k\pi}{10}\right); k = 1, 2, \ldots, 9.$

| | **List-I** | | **List-II** |
|---|---|---|---|
| **P.** | For each $z_k$ there exists *a* $z_j$ such that $z_k \cdot z_j = 1$ | **1.** | True |
| **Q.** | There exists *a* $k \in \{1, 2, \ldots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution $z$ in the set of complex numbers | **2.** | False |
| **R.** | $\dfrac{\|1-z_1\|\|1-z_2\|\ldots\|1-z_9\|}{10}$ equals | **3.** | 1 |
| **S.** | $1-\sum_{k=1}^{9}\cos\left(\dfrac{2k\pi}{10}\right)$ equals | **4.** | 2 |

| | **P** | **Q** | **R** | **S** |
|---|---|---|---|---|
| (A) | 1 | 2 | 4 | 3 |
| (B) | 2 | 1 | 3 | 4 |
| (C) | 1 | 2 | 3 | 4 |
| (D) | 2 | 1 | 4 | 3 |

**Solution**

For (P) in List I:

$$z_k = \cos\frac{2k\pi}{10} + i\sin\frac{2k\pi}{10}, \quad k = 1, 2, \cdots 9$$

Note $z_k \times z_{10-k} = e^{\frac{i2k\pi}{10}} \times e^{\frac{i2(10-k)\pi}{10}} = e^{\frac{i2\pi}{10}(k+10-k)}$

$$= e^{\frac{i2\pi}{10}\times 10} = \cos 2\pi + i\sin 2\pi = 1 + i(0) = 1$$

$\therefore\ Z_k \times Z_{10-k} = 1$

$$z_k \times z_{10-k} = 1$$
$$z_1 \times z_9 = 1$$
$$z_2 \times z_8 = 1$$
$$\vdots$$
$$z_9 \times z_1 = 1 \qquad \therefore P \to 1$$

For (Q) in List I:

$$z_1 \cdot z = z_k \Rightarrow z = \frac{z_k}{z_1} = \frac{e^{i2\frac{k\pi}{10}}}{e^{i2\frac{\pi}{10}}} = e^{i2(k-1)\frac{\pi}{10}} = z_{k-1}$$

$\therefore$ for $k = 1$, $z = z_0 = \cos 0 + i\sin 0 = 1$

for $k = 2$, $z = z_1$

for $k = 9$, $z = z_8$

$\therefore$ Solutions are these

$\therefore (Q) \to (2)$

For (R) in List I:

We know if 1, $z_1$, $z_2$, … $z_n$ are $n$, $n$th roots of unity them they are Roots of $z^n - 1 = 0$

$\therefore\ (z-1)(z-z_1)(z-z_2)\cdots(z-z_{n-1}) = (z^n - 1)$

$= (z-1)(z^{n-1} + z^{n-2} + \cdots z' + 1)$

$\Rightarrow\ (z-z_1)(z-z_2)\cdots(z-z_{n-1}) = z^{n-1} + 2^{n-2} + \cdots z' + 1$

Putting $z = 1$

$(1-z_1)(1-z_2)\cdots(1-z_{n-1}) = 1 + 1 + \cdots 1 = n$

$\therefore\ (1-z_1)(1-z_2)\cdots(1-z_9) = 10$

$\therefore\ |1-z_1||1-z_2|\cdots|1-z_9| = 10$

$$\Rightarrow\ \frac{|1-z_1||1-z_2|\cdots|1-z_9|}{10} = 1$$

$\therefore (R) \to (3)$

For (S) in List I:

We know $1 + \propto + \propto^2 \cdots + \propto^{n-1} = 0$ sum of $n$, $n$th Roots of 1

$z_0 + z_1 + z_2 + \cdots z_9 = 0$

$\therefore\ 1 - \{z_1 + z_2 + \cdots z_9\} = 1 - (-z_0) = 1 - (-1) = 2$

$\therefore (S) \to (4)$