**JEE ADVANCED 2014**
**PAPER 1**

**MATHEMATICS**

**One or More than One Options Correct Type**

This section contains **TEN** questions. Each has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct.

**1.** For every pair of continuous function $f,\ g:[0,1]\to\mathbb{R}$ such that

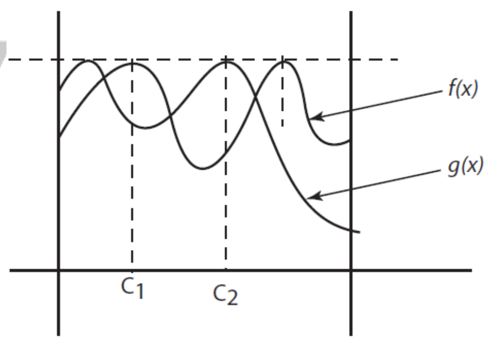
$$\max\{f(x) : x \in [0,1]\} = \max\{g(x) : x \in [0,1]\},$$

the correct statement(s) is (are):

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$
(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$

**Solution**

Suppose $f(x)$ is maximum at $c_1$ and $g(x)$ is maximum at $c_2$. When $f(x)$ is maximum $g(x)$ may or may not be maximum.

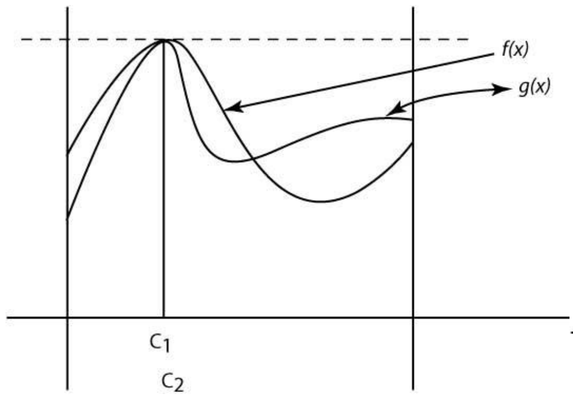


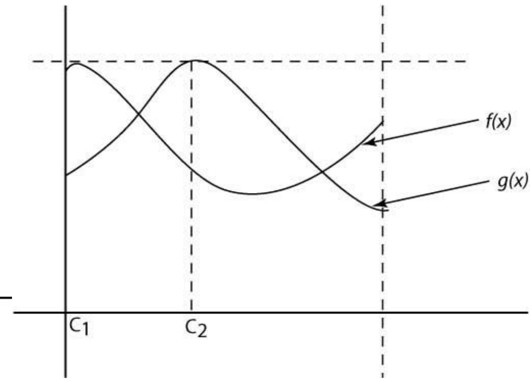

<AQ>'( )' should not be italicized<AQ>

$\therefore$ In the function $h(x) = f(x) - g(x)$.

we get $h(c_1) = f(c_1) - g(c_1) \geq 0$ and $h(c_2) = f(c_2) - g(c_2) \leq 0$

$\therefore$ $h(x) = 0$ for some $c \in [0, 1]$






$\therefore\ h(c) = 0$

$\Rightarrow f(c) - g(c) = 0$

$\therefore\ f(c) = g(c)$

**2.** A circle $S$ passes through the point (0, 1) and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then
(A) radius of $S$ is 8
(B) radius of $S$ is 7
(C) centre of $S$ is (–7, 1)
(D) centre of $S$ is (–8, 1)

**Solution**

Let the required circle is

$x^2 + y^2 + gx + 2fy + c = 0$ (1)

Now since (1) passes through (0, 1)

$\therefore\quad 0 + 1 + 0 + 2f + c = 0 \Rightarrow 2f + c + 1 = 0$ (2)

Since $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$ are other goal to (1) (or $x^2 + y^2 - 2x - 15 = 0$)

$\therefore\ 2g(-1) + 2f(0) = c - 15 \Rightarrow -2g = c - 15$ (3)

Also $2g(0) + 2f(0) = c - 1\ \therefore\ c = 1$ (4)

$\therefore$ From (2), (3) and (4) $f = -1$, $g = 7$ and $c = 1$

$\therefore$ Centre is (–7, 1) and radius = $\sqrt{7^2 + 1^2 - 1}$

$= \sqrt{49 + \cancel{1} - \cancel{1}}$
$= 7$



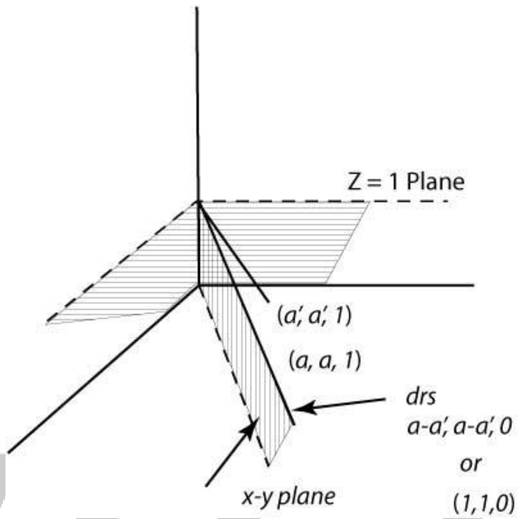
**3.** From a point $P(\lambda, \lambda, \lambda)$, perpendiculars $PQ$ and $PR$ are drawn respectively on the lines $y = x$, $z = 1$ and $y = -x$, $z = -1$. If $P$ is such that $\angle QPR$ is a right angle, then the possible value(s) of $\lambda$ is(are)
(A) $\sqrt{2}$

(B) 1
(C) –1

(D) $-\sqrt{2}$

**Solution**

Let ($a$, $a$, 1) be any point on the first line.
Similarly (–$b$, $b$, –1) is any point on the other line. (As line is the intersection of given planes)




$\therefore$ Direction ratios of first and second lines are respectively (1, 1, 0) and (−1, 1, 0).

Second line passes through (0, 0, −1)

Now since $PQ$ is $\perp$ to the first line.

$\therefore$ $(\lambda - a)\, 1 + (\lambda - a)\, 1 + (\lambda - 1)\, 0 = 0$

$\therefore$ $\lambda - a + \lambda - a = 0 \Rightarrow \lambda = a$

$\therefore$ $Q$ is $(a, a, 1)$

Also since $PR$ is $\perp$ to the second line.

$\therefore$ $(\lambda - (-b))(-1) + (\lambda - b)\, 1 + (\lambda - (-1))\, 0 = 0$

$\therefore$ $-\not{\lambda} - b + \not{\lambda} - b = 0$

$\Rightarrow b = 0$

$\therefore$ $R$ is $(0, 0, -1)$

Also since $PQ \perp PR$

$\therefore$ $(\lambda - a)(\lambda - 0) + (\lambda - a)(\lambda - 0) + (\lambda - 1)(\lambda + 1) = 0$

$\Rightarrow \lambda^2 = 1$, $\therefore \lambda = \pm 1$

Since $P$ and $Q$ are different points so $\lambda = 1$ is rejected. Hence $\lambda = -1$.

**4.** Let $\vec{x}$, $\vec{y}$ and $\vec{z}$ be three vector each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a nonzero vector perpendicular to $\vec{x}$ and $\vec{y}\times\vec{z}$ and $\vec{b}$ is a nonzero vector perpendicular to $\vec{y}$ and $\vec{z}\times\vec{x}$, then

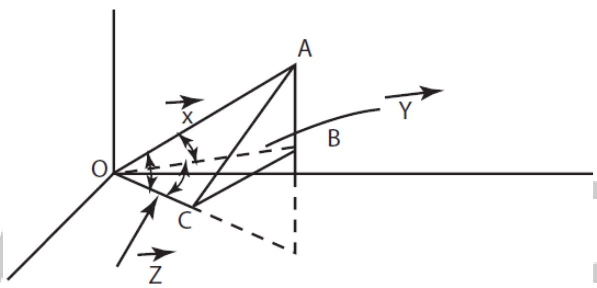
(A) $\vec{b} = (\vec{b}\cdot\vec{z})(\vec{z}-\vec{x})$

(B) $\vec{a} = (\vec{a}\cdot\vec{y})(\vec{y}-\vec{z})$

(C) $\vec{a}\cdot\vec{b} = -(\vec{a}\cdot\vec{y})|(\vec{b}\cdot\vec{z})$

(D) $\vec{a} = (\vec{a}\cdot\vec{y})(\vec{z}-\vec{y})$

**Solution**




$\angle AOB = \angle BOC = \angle CCOA = \frac{\pi}{3}$

According to question

$$\vec{a} = \lambda\{(\vec{x}\cdot\vec{z})\vec{y} - (\vec{x}\cdot\vec{y})\vec{z}\}$$
$$= \lambda\left\{\left(2\cos\frac{\pi}{3}\right)\vec{y} - \left(2\cos\frac{\pi}{3}\right)\vec{z}\right\} = \lambda(\vec{y}-\vec{z})$$

$$\because \vec{\alpha}\times(\vec{\beta}\times\vec{\gamma}) = (\vec{\alpha}\cdot\vec{\gamma})\vec{\beta} - (\vec{\alpha}\cdot\vec{\beta})\vec{\gamma}$$

$$\vec{b} = \mu\{(\vec{y}\cdot\vec{x})\vec{z} - (\vec{y}\cdot\vec{z})\vec{x}\}$$
$$= \mu\{\vec{z}-\vec{x}\}$$

Now $\vec{a}\cdot\vec{y} = \lambda\{2-1\}$ $\therefore \lambda = \vec{a}\cdot\vec{y}$

$\therefore$ $\vec{a} = \vec{a}\cdot\vec{y}(\vec{y}-\vec{z})$ (1)

Similarly $\vec{b} = \vec{b}\cdot\vec{z}(\vec{z}-\vec{x})$ (2)

Now $\vec{a}\cdot\vec{b} = (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})\{\vec{y}\cdot\vec{z} - \vec{y}\cdot\vec{x} - \vec{z}\cdot\vec{z} + \vec{z}\cdot\vec{x}\}$

$$= (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})\{1 - \not{1} - 2 + \not{1}\}$$

$$= -(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z}) \qquad (3)$$

$\therefore$ From (1), (2) and (3), we can conclude that the correct options are (A), (B) and (C).

**5.** Let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$ be given by $f(x)=(\log(\sec x+\tan x))^3$. Then

(A) $f(x)$ is an odd function

(B) $f(x)$ is a one-one function

(C) $f(x)$ is an onto function

(D) $f(x)$ is an even function

**Solution**

$$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$$

$$f(x)=\{\log(\sec x+\tan x)\}^3$$

$$f(-x)=\{\log(\sec x-\tan x)\}^3=\left\{\log\left(\frac{1}{\sec x+\tan x}\right)\right\}^3$$

$\because$ $\sec^2 x-\tan^2 x=1$

Now $f(x)+f(-x)=\{\log(\sec x+\tan x)\}^3+\{-\log(\sec x+\tan x)\}^3=0$

$\therefore$ $\underline{f(-x)=-f(x)}$ $\Rightarrow f(x)$ is odd. (1)

Also, $f'(x)=3\{\log(\sec x+\tan x)\}^2\dfrac{1}{\sec x+\tan x}\times(\sec x\tan x+\sec^2 x)$

$$=3\{\log(\sec x+\tan x)\}^2\sec x>0$$

$\because$ In $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\sec x$ is +ve.

Note $\sec x+\tan x=1\Rightarrow x=0\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

But at $x = 0$, $\log(\sec x+\tan x)=-\infty$

$\therefore$ $\{\log(\sec x + \tan x)\}^2 > 0$

$\therefore$ $f(x)$ is strictly increasing in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (2)

Now when $x \to \frac{\pi}{2}^-$ , $f(x) \to \infty$

and when $\pi \to \frac{-\pi}{2}^+$ , $f(x) \to -\infty$

$\therefore$ $f(x)$ being continuous in its domain, it covers whole codomain i.e. $\mathbb{R}$ .

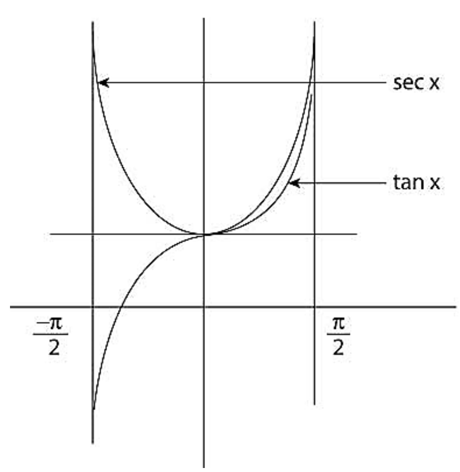
$\therefore$ It is onto (3)

**Note:** $\lim\limits_{x \to \frac{\pi}{2}^-} (\sec x + \tan x)$

$$= \lim_{h \to 0} \left( \sec\left(\frac{+\pi}{2} - h\right) + \tan\left(\frac{\pi}{2} - h\right) \right)$$
$$= \lim_{h \to 0} (\operatorname{cosec} h + \cot h)$$
$$= \frac{1 + \cosh}{\sinh} = +\infty$$

$$\lim_{x \to \frac{-\pi}{2}^+} (\sec x + \tan x)$$

$$= \lim_{h \to 0} \left\{ \sec\left(\frac{-\pi}{2} + h\right) + \tan\left(\frac{-\pi}{2} + h\right) \right\}$$
$$= \lim_{h \to 0} \left( \operatorname{cosec}\left(\frac{\pi}{2} - h\right) - \tan\left(\frac{\pi}{2} - h\right) \right)$$
$$= \lim_{h \to 0} (\sec h - \cot h)$$
$$= \lim_{h \to 0} \left( \frac{1 - \cos h}{\sin h} \right) = \frac{\cancel{2} \sin^{\cancel{2}} \frac{h}{2}}{\cancel{2} \sin \cancel{\frac{h}{2}} \cos \frac{h}{2}}$$
$$= \lim_{h \to 0} \tan \frac{h}{2} = 0$$



Therefore from (1), (2) and (3) we can conclude that the correct options are (A), (B) and (C).

**6.** Let $f : [a, b] \to [1,\infty)$ be a continuous function and let $g:\mathbb{R}\to\mathbb{R}$ be defined as

$$g(x)=\begin{cases} 0 & \text{if } x<a, \\ \int_a^x f(t)dt & \text{if } a\le x\le b, \\ \int_a^b f(t)dt & \text{if } x>b \end{cases} \text{ Then}$$

(A) $g(x)$ is continuous but not differentiable at $a$

(B) $g(x)$ is differentiable on $\mathbb{R}$

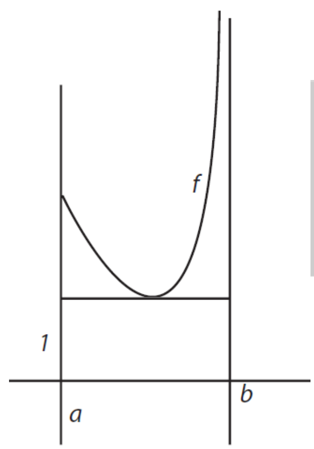
(C) $g(x)$ is continuous but not differentiable at $b$

(D) $g(x)$ is continuous and differentiable at either $a$ or $b$ but not both

**Solution**

Checking continuity of $g$.






$$\underset{h\to 0}{g\,(a-h)}=0$$

$$\underset{h\to 0}{g\,(a+h)}=\underset{h\to 0}{\int_a^{a+h} f(t)\,dt}=\int_a^a f(t)dt$$

$$g(a)=\int_a^a f(t)dt=0$$

$\therefore$ $g$ is continuous at $a$.

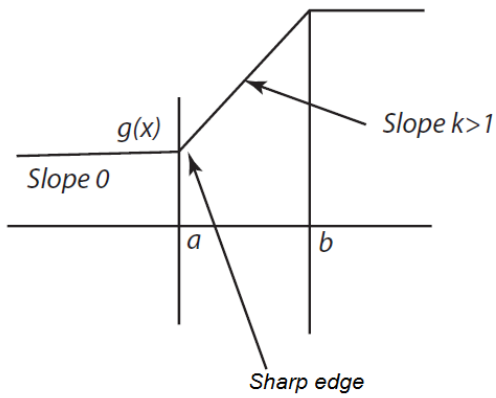
Similarly $g$ is continuous at $b$.

As $$\underset{h\to 0}{g\,(b-h)}=g(b)=\underset{h\to 0}{g\,(b+h)}=\int_a^b f(t)dt$$

Now

$$g'(x) = \begin{cases} 0, & x < a \\ f(x), & a < x < b \\ 0, & x > b \end{cases}$$

Since $f(x) \geq 1$ in $[a, b]$ given, so as we cross $a$ and $b$ according to $g(x)$ function, there are sharp edges encountered due to abrupt change in the slopes from 0 to $k$ and then from $k$ to 0, where $k \geq 1$.




$g'(a+) = f(a+h) \geq 1$ etc.



**7.** Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then

(A) $f(x)$ is monotonically increasing on $[1, \infty)$

(B) $f(x)$ is monotonically decreasing on $(0, 1)$

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

(D) $f(2^x)$ is an odd function of $x$ on $\mathbb{R}$

**Solution**

$$f(x) = \int_{\frac{1}{x}}^{x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$$

$$\therefore \frac{d}{dx} f(x) = \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} \frac{d}{dx} x - \frac{e^{-\left(\frac{1}{x}+x\right)}}{\frac{1}{x}} \times \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + xe^{-\left(x+\frac{1}{x}\right)} \times \left(-\frac{1}{x^2}\right)$$

$$= \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{1}{x} e^{-\left(x+\frac{1}{x}\right)}$$

$$= \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x} > 0$$

$\therefore$ $f(x)$ is strictly increasing in $(0, \infty)$ (1)

$$\text{Now } f(x) + f\left(\frac{1}{x}\right) = \int_{\frac{1}{x}}^{x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt + \int_{x}^{\frac{1}{x}} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$$

$$= \int_{\frac{1}{x}}^{\frac{1}{x}} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt = 0 \qquad (2)$$

$$\text{Now } f(2^x) + f\left(\frac{1}{2^x}\right) = f(2^x) + f(2^{-x})$$

$$= 0$$

$\therefore$ $f(2^x)$ is an odd function (3).

**Note:** Let $2^x = \mu$ $\quad \therefore \log_2\mu = x$

$\therefore$ For $\mu \in (0, \infty)$, $\quad x \in (-\infty, \infty)$

$\therefore$ we can say $f(2^x) = h(x)$ an odd function $\because$ $h(-x) = -h(x)$

$\therefore$ From (1), (2) and (3), we can conclude that the correct options are (A), (C) and (D).

**8.** Let $M$ be a $2 \times 2$ symmetric matrix with integer entries. Then $M$ is invertible if

(A) The first column of $M$ is the transpose of the second row of $M$
(B) The second row of $M$ is the transpose of the first column of $M$
(C) $M$ is a diagonal matrix with nonzero entries in the main diagonal
(D) The product of entries in the main diagonal of $M$ is not the square of an integer

**Solution**

$$\text{Let } M = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \qquad a, b, d \in I$$

(A) $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} \Rightarrow a = b = d \quad \Rightarrow |M| = \begin{vmatrix} a & a \\ a & a \end{vmatrix} = 0$

$\Rightarrow$ Not invertible. $\therefore$ (A) is false

(B) $\begin{bmatrix} b & d \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix} \Rightarrow a = b = d$

$\Rightarrow |M| = 0 \Rightarrow$ Not invertible. $\therefore$ (B) is false

(C) If M is diagonal matrix then $M = \begin{bmatrix} a & o \\ o & d \end{bmatrix} \Rightarrow |M| = ad \neq 0$

$\Rightarrow$ M invertible. $\therefore$ (C) is correct

(D) Given $-ad \neq b^2$

Now $|M| = ad - b^2 \neq 0$ for $M$ to be invertible.

$\therefore$ (D) is True

**9.** Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^5 - 5x + a$$

Then

(A) $f(x)$ has three real roots if $a > 4$

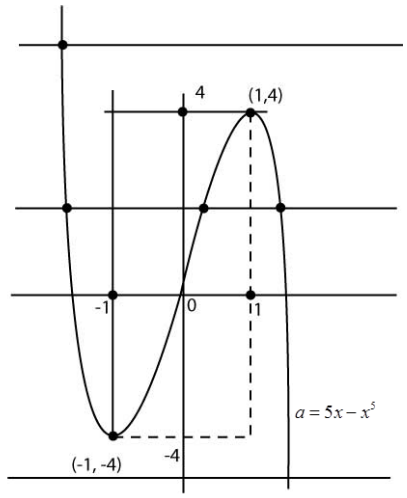
(B) $f(x)$ has only one real root if $a > 4$

(C) $f(x)$ has three real roots if $a < -4$

(D) $f(x)$ has three real roots if $-4 < a < 4$

**Solution**

$f(x) = x^5 - 5x + a$ there will be different polynomials, depending on the parameter '$a$'. Now for roots of each of these in general $f(x) = 0$




*i.e.* $a = 5x - x^5 = x(5 - x^5)$

$\therefore$ Parameter $a$ is a function of $x$

i.e. $a(x) = x\,(5 - x^5)$

Now $a'(x) = 5 - 5x^4$

$\therefore$ Extrema occur at $a'(x) = 0$

i.e., when $x^4 = 1$ or $x = 1$ and $x = -1$ (only real roots considered)

$\therefore$ $a''(x) = -20x^3$

$a''(1) < 0$ Max.

$a''(-1) > 0$ Min

$\therefore$ Max value = $a(1) = 4$

Min value = $a(-1) = -4$

$\therefore$ When $-4 < a < 4$, there are three points i.e., $x$ values where $f(x) = 0$, i.e. 3 roots of $f(x)$ for any value of $a$ lying in $(-4, 4)\}$. (1)

When $|a| > 4$, there is only one $x$ for which $f(x) = 0$ (2)

Hence from (1) and (2) we can conclude that (B) and (D) are correct options.

**10.** Let $M$ and $N$ be two $3 \times 3$ matrices such $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(A) Determinant of $(M^2 + MN^2)$ is 0

(B) There is a $3 \times 3$ non-zero matrix $U$ such that $(M^2 + MN^2)U$ is the zero matrix.

(C) Determinant of $(M^2 + MN^2) \geq 1$

(D) For a $3 \times 3$ matrix $U$, if $(M^2 + MN^2)U$ equals the zero matrix then $U$ is the zero matrix

**Solution**

Given $MN = NM$ $\therefore$ $a^2 - b^2 = (a + b)(a - b)$ of algebra of numbers is applicable

$\therefore$ Now $M^2 = N^4$

$\Rightarrow M^2 - N^4 = 0$ (Null matrix)

$\Rightarrow (M + N^2)(M - N^2) = 0$

Now since $M \neq N^2$ (Given)

$\therefore$ the possibilities are,

$(M + N^2) = 0$ and $M - N^2 \neq 0$ (1)

or $(M + N^2) \neq 0$ and $M + N^2 \neq 0$ (2)

Now we know, if $A$ and $B$ are non–null square matrix and $AB = 0$ then $A$ and $B$ both are singular i.e. $|A| = 0$ and $|B| = 0$ and $AB = 0$

**Note:** E.g. let $A$ is non–singular,

$\therefore$ $B = \text{In}(B) = A^{-1}AB = 0$ $\because$ $AB = 0$ assumed

$\therefore$ $B$ is singular, which is a contradiction.

$\therefore$ $A$ has to be singular. Similarly then $B$ also has to be singular.

$\therefore$ From (1) and (2) we conclude the only possibility is $|M + N^2| = 0$

Now checking options:

(A) $|M^2 + MN^2| = |M||M + N^2| = 0$

$\therefore$ (A) is correct

(B) $(M^2 + MN^2)\,U = 0$

$\because$ $M^2 + MN^2$ is singular

$\therefore$ $U$ has infinitely many possible values (Non–trivial solutions)

$\therefore$ (B) is true

(C) False $\because |M^2 + MN^2| = 0$

(D) $\because |M^2 + MN^2| = 0$

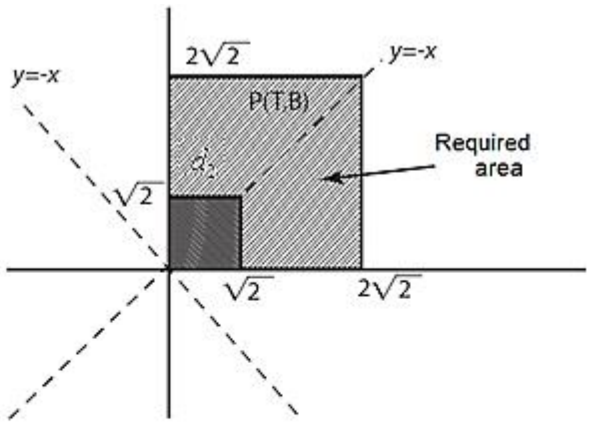
$\therefore$ $U$ is not a necessarily a zero matrix

**Integer Answer Type**

This section contains **TEN** questions. The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive.

**11.** For a point $P$ in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point $P$ from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region $R$ consisting of all points $P$ lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is _____.

**Solution**




$$d_1 = \frac{|x-y|}{\sqrt{2}}$$

$$d_2 = \frac{|x+y|}{\sqrt{2}}$$

$\therefore$ According to the question

$$2 \le \frac{|x-y|}{\sqrt{2}} + \frac{|x+y|}{\sqrt{2}} \le 4$$

$$\Rightarrow\ 2\sqrt{2} \le |x-y| + |x+y| \le 4\sqrt{2} \qquad (1)$$

Since $x, y \ge 0$ in the first quadrant

When $x > y$ (or $y - x < 0$),

$|x - y| = x - y$ and $|x + y| = x + y$

$\therefore$ (1) is true given that, $2\sqrt{2} \le x - y + x + y \le 4\sqrt{2} \ \Rightarrow\ \sqrt{2} \le x \le 2\sqrt{2}$

Checking with (2, 1) in region $x > y$ i.e. $2 > 1$

$\therefore$ We shade area below $y = x$ from $\left[\sqrt{2}, 2\sqrt{2}\right]$

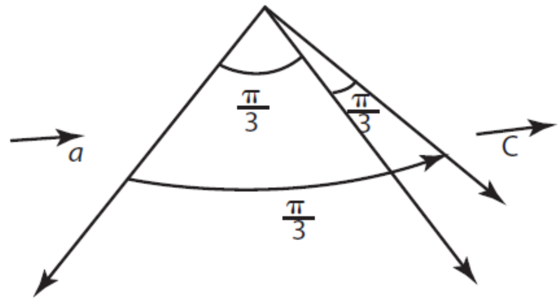
Area of this region $= \frac{1}{2}\left(2\sqrt{2} \times 2\sqrt{2}\right) - \frac{1}{2}\sqrt{2} \times \sqrt{2} = 4 - 1 = 3$

By symmetry about $y = x$, total area required = 6

**12.** Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, where $p$, $q$ and $r$ are scalars, then the values of $\dfrac{p^2 + 2q^2 + r^2}{q^2}$ is _____.

**Solution**

Given $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$ (1)




Taking dot product with $\vec{a}$.

$$\therefore 0+\vec{a}\cdot\vec{b}\times\vec{c}=p(1.1.\cos 0)+q\left(1.1.\cos\frac{\pi}{3}\right)+r\left(1.1.\cos\frac{\pi}{3}\right)$$

$$\Rightarrow \vec{a}\cdot\vec{b}\times\vec{c}=p+\frac{q}{2}+\frac{r}{2} \quad (2)$$

Taking the dot product of (1) with $\vec{b}$.

$$0+0=\frac{p}{2}+q+\frac{r}{2} \quad (3)$$

Taking the dot product of (1) with $\vec{c}$

$$\vec{c}\cdot\vec{a}\times\vec{b}+0=\frac{p}{2}+\frac{q}{2}+r \quad (4)$$

From (2) and (4)

$$p+\frac{q}{2}+\frac{r}{2}=\frac{p}{2}+\frac{q}{2}+r$$

$$\frac{p}{2}=\frac{r}{2}\Rightarrow p=r$$

Now from (3) $0=\frac{r}{2}+q+\frac{r}{2}$

$\Rightarrow q=-r$

Now $\dfrac{p^2+2q^2+r^2}{q^2}=\dfrac{r^2+2(-r)^2+r^2}{(-r)^2}=\dfrac{4r^2}{r^2}=4$

**13.** The largest value of the non-negative integer $a$ for which

$$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$$

is _____.

**Solution**

$$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$$

$$\Rightarrow \lim_{x\to 1}\left\{\frac{\sin(x-1)-a(x-1)}{\sin(x-1)+(x-1)}\right\}^{(1-\sqrt{x})}=\frac{1}{4}$$

$$\Rightarrow \lim_{x\to 1}\left\{\frac{\frac{\sin(x-1)}{x-1}-a}{\frac{\sin(x-1)}{x-1}+1}\right\}^{1+\sqrt{x}} = \frac{1}{4}$$

$$\Rightarrow \left\{\frac{1-a}{1+1}\right\}^{1+1} = \frac{1}{4}$$

$$\Rightarrow \left(\frac{1-a}{2}\right)^{2} = \frac{1}{4}$$

$$\therefore\ \frac{1-a}{2} = \pm\frac{1}{2}$$

$\therefore\ 1 - a = \pm 1 \Rightarrow a = 0, 2$

$\therefore$ Largest value = 2

**14.** Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation

$$f(x) = \frac{10-x}{10}$$

is ______.

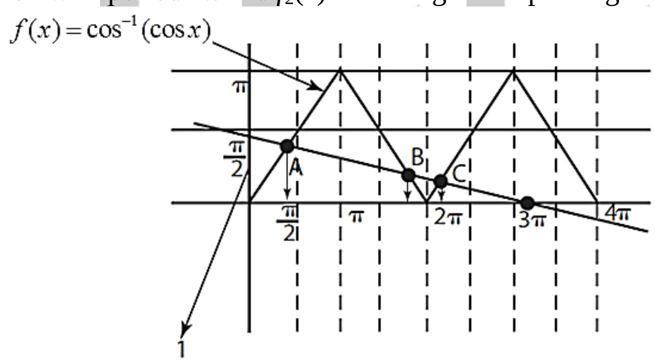
**Solution**

We need to find the point of intersection of the curves

$f_1(x) = \cos^{-1}(\cos x)$ and $f_2(x) = \dfrac{10-x}{10}$

in the domain $[0, 4\pi]$.

$f_1(x)$ is a period function with period $2\pi$ and $f_2(x)$ is a straight line plotting both graphs.




$\therefore$ $A$, $B$ and $C$ are the points of intersection of both curves which obviously satisfy the given equations, hence there are three such points.

**15.** Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h:\mathbb{R}\to\mathbb{R}$ by

$$h(x) = \begin{cases} \max \quad \{f(x), g(x)\} & \text{if } x \le 0, \\ \min \quad \{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$

The number of points at which $h(x)$ is not differentiable is ______.

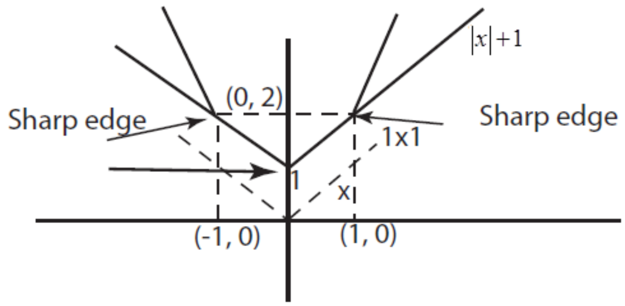
**Solution**

$f:\mathbb{R}\to\mathbb{R}$ $\qquad\qquad$ $g:\mathbb{R}\to\mathbb{R}$

$f(x) = |x| + 1$ $\qquad\qquad$ $g(x) = x^2 + 1$

$h : \mathbb{R} \to \mathbb{R}$

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \le 0, \\ \min\{f(x), g(x)\} & \text{if } x > 0 \end{cases}$$




Points of Intersection are, (1, 2) and (−1, 2).

$h(x) = x^2 + 1$, in $(-\infty, -1)$

$x^2 + 1 = |x| + 1$ at $-1$

$|x| + 1$, in $(-1, 0)$

$|x| + 1 = x^2 + 1$ at 0

$\quad = x^2 + 1$, in (0, 1)

$x^2 + 1 = |x| + 1$ at 1

$\quad = |x| + 1$, in $(1, \infty)$

At sharp edges i.e. at −1, 0 and 1, there is no smooth turn, so no derivative exits there. Elsewhere, function is continuous and derivative. Hence there are three such points.

**16.** Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is ______.

**Solution**

$n_1 < n_2 < n_3 < n_4 < n_5$

$n_1 = n_1$ −−−−−−−*

$$n_2 = \underbrace{n_1 + \propto_1 + 1} \qquad (1)$$

$$n_3 = n_2 + \propto_2 + 1 = \overbrace{(n_1 + \propto_1 + 1)} + \propto_2 + 1 = \underbrace{n_1 + \propto_1 + \propto_2 + 2} \qquad (2)$$

$$n_4 = n_3 + \propto_3 + 1 = \overbrace{n_1 + \propto_1 + \propto_2 + 2} + \propto_3 + 1 = \underbrace{n_1 + \propto_1 + \propto_2 + \propto_3 + 3} \qquad (3)$$

$$n_5 = n_4 + \propto_4 + 1 = \overbrace{n_1 + \propto_1 + \propto_2 + \propto_3 + 3} + \propto_4 + 1$$
$$= n_1 + (\propto_1 + \propto_2 + \propto_3 + \propto_4) + 4 \qquad (4)$$

Adding (1), (2), (3), and (4)

$20 = 5n_1 + 4\propto_1 + 3\propto_2 + 2\propto_3 + \propto_4 + 10$

Let $n_1 = \propto_0 + 1$

$\therefore\ 20 = 5(\propto_0 + 1) + 4\propto_1 + 3\propto_2 + 2\propto_3 + \propto_4 + 10$

$\Rightarrow 5\propto_0 + 4\propto_1 + 3\propto_2 + 2\propto_3 + \propto_4 = 5 \qquad (5)$

where $\propto_0, \propto_1, \propto_2, \propto_3, \propto_4 \ge 0$ and integers are values which are introduced to adjust the gaps in the numbers $n_1$, $n_2$, $n_3$, $n_4$ and $n_5$.

Now the problem transforms into finding non-negative integral solutions of (5).

$\therefore$ Numbers of solutions is equal to the coefficient of

$x^5$ in $\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^2}\right)$.

= coeff. of $x^5$ in $(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}$

= coeff. of $x^5$ in $\underbrace{(1+x+x^2+x^3+x^4+x^5+\cdots)(1+x^2+x^4+x^6+\cdots)}$

= coeff. of $x^5$ in $\underbrace{(1+x^3+x^6+x^9+\cdots)(1+x^4+x^8+\cdots)}(1+x^5+\cdots)$

Note -{terms having powers more than 5 are not required.}

= coeff. of $x^5$ in $(1 + x^2 + x^4 + x + x^3 + x^5 + x^2 + x^4 + x^3 + x^5 + x^4 + x^5)(1 + x^4 + x^3)(1 + x^5)$

= coeff. of $x^5$ in $(1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5)(1 + x^5 + x^4 + x^3)$

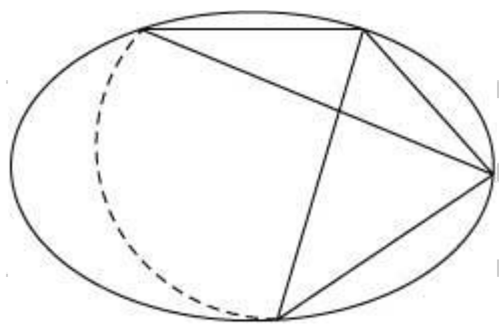
= 1 + 1 +2 +3 = 7

Note: Finding the number of solutions in (5) can be understood as making a sum 5 by choosing tickets numbered 1, 2, 3, 4, 5 taking none or more tickets.

**17.** Let $n \geq 2$ be an integer. Take $n$ distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of $n$ is _____.

**Solution**

According to question



<AQ>Draw this as a circle<AQ>

Number of blue lines = $n$ = number of sides of polygon so formed.

Number of red lines = ${}^nC_2 - n$

$\because$ By joining $n$ points (not more than 2 on a line) there are ${}^nC_2$ lines formed because for each line two points are required.

Also red lines come after excluding sides of polygon.

$\therefore$ $n = {}^nC_2 - n$ or ${}^nC_2 = 2n$

or $\frac{\cancel{n}(n-1)}{2} = 2\cancel{n}$ or $n - 1 = 4$ $\because n \neq 0$

$\therefore$ $n = 5$

**18.** The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is _____.

**Solution**

For slope, we differentiate the equation

$\therefore$ $2(y-x^5)'\left(\frac{dy}{dx} - 5x^4\right) = x\{2(1+x^2)2x\} + (1+x^2)^2 1$

Now putting (1, 3) in it,

$$2(3-1)\left(\frac{dy}{dx}-5\right)=1\{2(2)2\}+(1+1)^2$$

$$\Rightarrow 4\left(\frac{dy}{dx}-5\right) = 8 + 4 \Rightarrow \frac{dy}{dx}=8$$

$\therefore$ Slope at (1, 3) is 8.

**19.** Let $a$, $b$, $c$ be positive integers such that $\frac{b}{a}$ is an integer. If $a$, $b$, c are in geometric progression and the arithmetic mean of $a$, $b$, $c$, is $b$ + 2, then the value of $\frac{a^2+a-14}{a+1}$ is _____.

**Solution**

Let $a = a$

$b = ar$

$c = ar^2$

where $r$ is integer

$\because \frac{b}{a}$ is an integer.

Now according to the question,

$$\frac{a+ar+ar^2}{3}=ar+2$$

$\because$ (A·M) = ($b$ + 2)

$\therefore a + ar + ar^2 = 3ar + 6$

$\Rightarrow ar^2 - 2r + a = 6$

$$\Rightarrow \underbrace{r^2-2r+1}_{\text{integer}} = \underbrace{\frac{6}{a}}_{\text{integer}} \qquad (1)$$

$$\Rightarrow (r-1)^2=\frac{6}{a}$$

If $a$ = 1, 2, 3, 4, 5, 6 is not a perfect square and integer.

$\therefore$ Only possibility is that $a$ = 6.

$$\therefore \frac{a^2+a-14}{a+1}=\frac{36+6-14}{6+1}=\frac{284}{7}=4$$

**20.** The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}dx$ is _____.

**Solution**

$$\int_0^1 \underset{\text{I}}{4x^3}\underbrace{\frac{d^2}{dx^2}(1-x^2)^5}_{\text{II}}dx$$

Integrating by parts

$$I=4x^3\left[\frac{d}{dx}(1-x^2)^5\right]_0^1-\int_0^1 12x^2\frac{d}{dx}\left(1-x^2\right)^5dx$$

$$=4x^3[5(1-x^2)^4(-2x)]_0^1-12\left[[x^2(1-x^2)^5]_0^1-\int_0^1 2x(1-x^2)^5dx\right.$$

$= 0 - 0 + 12\int_0^1 2x(1-x^2)^5\,dx$

Now putting $1 - x^2 = t$, $\therefore -2xdr = dt$

$\therefore\ I = -12\int_1^0 t^5 dt$

$\therefore$ where $x = 0$, $t = 1$
Where $x = 1$, $t = 0$

$$I = 12\times\int_0^1 t^5 dt = 12\times\left[\frac{t^6}{6}\right]_0^1 = 12\times\frac{1}{6} = 2$$