JEE ADVANCED | 2016
PAPER 1

MATHEMATICS

Single Option Correct Type

This section contains 5 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

1. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

(A) 380 (B) 320
(C) 260 (D) 95

Solution

(A) The club consists of 6 girls and 4 boys.

If a team of 4 members to be selected which consists at most 1 boy (including 1 captain), then the number of ways of selecting the team is obtained as follows:

$$^4C_1(^4C_1 \cdot {^6C_3} + {^6C_4}) = 4(80 + 15) = 380 \text{ ways}$$

2. The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$
(C) $\frac{1}{27}$ (D) $\frac{1}{25}$

Solution

(C) It is given that

$$4\alpha x^2 + \frac{1}{x} \geq 1 \quad \forall x > 0$$

That is, $4\alpha x^2 \geq 1 - \frac{1}{x}$

$$4\alpha \geq \left(\frac{1}{x^2} - \frac{1}{x^3}\right) \forall\, x > 0$$

Now, let us consider that

$$f(x) = \frac{1}{x^2} - \frac{1}{x^3}$$

Therefore, $f'(x) = \frac{-2}{x^3} + \frac{3}{x^4} = 0$

When $x = 3/2$:

$$(4\alpha) \geq \left(\frac{1}{x^2} - \frac{1}{x^3}\right)$$

$$4\alpha \geq \left(\frac{4}{9} - \frac{8}{27}\right)$$

That is, $\alpha \geq \frac{1}{27}$

and hence the least value of α is

$\alpha_{\text{least}} = \frac{1}{27}$

3. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\vartheta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\vartheta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

(A) $2(\sec\vartheta - \tan\vartheta)$ (B) $2\sec\vartheta$

(C) $-2\tan\vartheta$ (D) 0

Solution

(C) We have $\theta \in \left(\frac{-\pi}{6}, \frac{-\pi}{12}\right)$.

- It is given that α_1 and β_1 are the roots of the equation

$$x^2 - 2x\sec\vartheta + 1 = 0$$

That is, $\alpha_1, \beta_1 = \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 1}}{2}$

$$\alpha_1, \beta_1 = \sec\vartheta \pm \tan\vartheta (\because \sec\vartheta > 0 \text{ and } \tan\vartheta < 0)$$

Since it is given that $\alpha_1 > \beta_1$, we get

$$\alpha_1 = \sec\vartheta - \tan\vartheta$$

- It is also given that α_2 and β_2 are the roots of the equation

$$x^2 + 2x\tan - 1 = 0$$

That is, $\alpha_2, \beta_2 = \frac{-2x\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$

$$\alpha_2, \beta_2 = -\tan\vartheta \pm \sec\vartheta$$

Since it is given that $\alpha_2 > \beta_2$, we get

$$\alpha_2 = -\tan\vartheta + \sec\vartheta$$

and $\beta_2 = -\tan\vartheta - \sec\vartheta$

Therefore,

$$\alpha_1 + \beta_2 = \sec\vartheta - \tan\vartheta - \tan\vartheta - \sec\vartheta = -2\tan\vartheta$$

4. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation in the set S is equal to

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$

(C) 0 (D) $\frac{5\pi}{9}$

Solution

(C) Let us consider

$S = \left\{ x \in (-\pi, \pi), x \neq 0, \pm \frac{\pi}{2} \right\}$

The given equation is

$$\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$$

$$\Rightarrow \frac{\sqrt{3}}{\cos x} + \frac{1}{\sin x} + 2\left(\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}\right) = 0$$

$$\Rightarrow \sqrt{3}\sin x + \cos x + 2(\sin^2 x - \cos^2 x) = 0$$

$$\Rightarrow \sqrt{3}\sin x + \cos x = 2\cos 2x$$

$$\Rightarrow \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \cos 2x$$

$$\Rightarrow \cos x \cos\frac{\pi}{3} + \sin x \sin\frac{\pi}{3} = \cos 2x$$

$$\Rightarrow \cos 2x = \cos\left(x - \frac{\pi}{3}\right)$$

$$\Rightarrow 2x = 2n\pi \pm \left(x - \frac{\pi}{3}\right) \ (x \in \mathrm{I})$$

- Case 1: When $2x = 2n\pi + x - \frac{\pi}{3}$, we have $x = 2n\pi - \frac{\pi}{3}$.

  If ***n* =** 0, we get $x = -\frac{\pi}{3}$.

  If ***n* =** 1, we get $x = 2\pi - \frac{\pi}{3}$.

  If ***n* =** −1, $x = -2\pi - \frac{\pi}{3}$.

- Case 2: When $2x = 2n\pi - x + \frac{\pi}{3}$, we get $x = \frac{2n\pi}{3} + \frac{\pi}{9}$.

  If ***n* =** 0, we get $x = \frac{\pi}{9}$.

  If ***n* =** 1, we get $x = \frac{2\pi}{3} + \frac{\pi}{9}$.

  If ***n* =** 2, we get $n = 2x = \frac{4\pi}{3} + \frac{\pi}{9}$.

  If ***n* =** −1, we get $x = \frac{-2\pi}{3} + \frac{\pi}{9}$**.**

Therefore, the sum of all distinct solutions of the given equation is

$\frac{-\pi}{3} + \frac{\pi}{9} + \frac{2\pi}{3} + \frac{\pi}{9} - \frac{2\pi}{3} + \frac{\pi}{9} = 0$

5. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that *P* (computer turns out to be defective given that is produced in plant T_1) = -10*P* (computer turns out to be defective given that it is produced in plant T_2), where *P*(*E*) denotes the probability of an event *E*. A computer produces in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$ (C) $\frac{78}{93}$ (D) $\frac{75}{83}$

Solution

(C) Let P_1 be the defective computers that are produced from plant T_1 and P_2 be that from plant T_2.

The total percentage of the defective computers produced is 7%.

Now, $P_1 = 10P$ and $P_2 = P$.

The computers produced that are defective:

$$\frac{20}{100}\times P_1+\frac{80}{100}\times P_2=\frac{7}{100}$$

$$20P_1 + 80P_2 = 7$$

$$200P + 80P = 7$$

$$P=\frac{7}{280}=\frac{1}{40}$$

Now, the probability of the defective products is calculated as follows:

$$\frac{20}{100}P_1+\frac{80}{100}P_2=\frac{20}{100}\times\frac{1}{4}+\frac{80}{100}\times\frac{1}{40}=\frac{28}{400}=\frac{7}{100}$$

The probability of producing **NOT** defective computers is

$$1-\frac{7}{100}=\frac{93}{100}$$

The probability that plant T_2 produces **NOT** defective computers is calculated as follows:

$$\frac{(80/100)\times(39/40)}{(93/100)}=\frac{78}{93}$$

One or More Than One Option Correct Type

This section contains 8 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONE OR MORE** is(are) correct.

6. A solution curve of the differential equation

$$(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0,\ x > 0$$

passes through the point (1, 3). The solution curve

(A) intersects $y = x + 2$ exactly at one point.

(B) intersects $y = x + 2$ exactly at two points.

(C) intersects $y = (x + 2)^2$.

(D) does NOT intersect $y = (x + 3)^2$.

Solution

(A), (D) The given differential equation is

$$[x^2+4x+4+y(x+2)]\frac{dy}{dx}-y^2=0\ (x>0)$$

which is further simplified as follows:

$$[(x+2)^2+y(x+2)]\frac{dy}{dx}-y^2=0$$

Substituting $x+2=t$, we get

$$\frac{dx}{dy}=\frac{dt}{dy}$$

Now,

$$(x+2)^2+y(x+2)-y^2\frac{dx}{dy}=0$$

That is,

$$t^2 + yt - y^2\frac{dt}{dy} = 0$$

$$y^2\frac{dt}{dy} - yt - t^2 = 0$$

$$\frac{1}{t^2}\frac{dt}{dy} - \frac{1}{yt} = \frac{1}{y^2}$$

Let $\frac{1}{t} = z$; therefore,

$$\frac{dt}{dy}\left(-\frac{1}{t^2}\right) = \frac{dz}{dy}$$

Now, $\frac{-dz}{dy} - \frac{z}{y} = \frac{1}{y^2} \Rightarrow \frac{dz}{dy} + \frac{z}{y} = \frac{-1}{y^2}$

$$\Rightarrow d(zy) = \int -\frac{1}{y}dy$$

$$\Rightarrow zy = -\ln|y| + \ln c$$

$$\Rightarrow \frac{y}{t} = -\ln|y| + \ln c$$

$$\Rightarrow \frac{y}{(x+2)} = -\ln|y| + \ln c \quad (1)$$

which passes through the point (1, 3). Therefore, from Eq. (1), we get

$$\frac{z}{3} = -\ln 3 + c \Rightarrow c = \ln 3e$$

$$\frac{y}{x+2} = -\ln|y| + \ln 3e = \ln\left(\frac{3e}{|y|}\right)$$

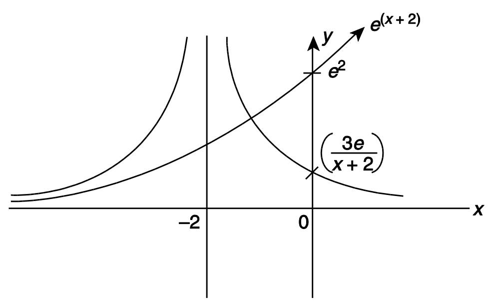
$$\frac{3e}{|y|} = e^{y/(x+2)}$$

$$3e = |y|e^{y/(x+2)}$$

Substituting $y = (x + 2)$, we get $3e = |x + 2|e^1$

$$|x + 2| = 3 \Rightarrow x + 2 = -3, 3 \Rightarrow x = -5, 1$$

Therefore, $x = 1$ (since $x \neq -5$).



That is, the solution curve intersects $y = (x + 2)$ exactly at one point and **NOT** at two points. Therefore, option (A) is correct and option (B) is incorrect.

Checking for option (C): We have

$$\frac{3e}{|(x+2)^2|} = e^{(x+2)}$$

which meets at two points for $x < 0$ and for $x > 0$, there is no intersection point.

Hence, option (C) is incorrect.

Checking for option (D): We have

$$\frac{3e}{(x+3)^2} = e^{\frac{(x+3)^2}{(x+2)}} = e^{\frac{(x+2)^2+1+2(x+2)}{(x+2)}} = e^{2+\frac{1}{(x+2)}+(x+2)}$$

Therefore, there is no intersection point for $x > 0$.
Hence option (D) is correct.

7. Consider a pyramid OPQRS located in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP = 3. The point S is directly above the midpoint T of diagonal OQ such that TS = 3. Then

(A) the acute angle between OQ and OS is $\pi/3$.

(B) the equation of the plane containing the triangle OQS is $x - y = 0$.

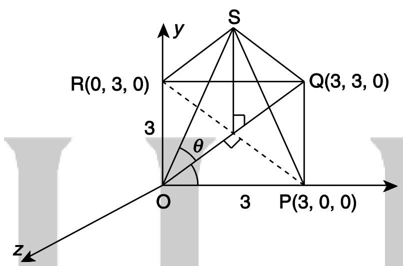
(C) the length of the perpendicular from P to the plane containing the triangle OQS is $3/\sqrt{2}$.

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{15/2}$.

Solution

(B), (C), (D)

Let us consider a pyramid OPQRS in the first octant with O as origin as shown in the following figure:



Now, the midpoint of OQ is $T\left(\frac{3}{2},\frac{3}{2},0\right)$.

Also, it is obvious that the point S is given by $S\left(\frac{3}{2},\frac{3}{2},3\right)$.

Now, $OT = \sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2+0} = \frac{3}{2}\sqrt{2}$

Therefore, $\cos\theta = \frac{3\sqrt{2}}{2\ 3} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

where ϑ is the angle between OQ and OS, which is calculated as follows:
The ratio of the plane containing triangle ΔOQS can be written as

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 3 & 0 \\ 3/2 & 3/2 & 3 \end{vmatrix} = i(9) - i(9) + k(0) = i(i - j)$$

Therefore, the ratio is (1, −1, 0).
Now, the equation of the plane is

$$1(x-0) - 1(y-0) + 0(z-0) = 0$$
$$x - y = 0$$

Hence, option (B) is correct.
The coordinates of the length of perpendicular from point P to the plane $x - y = 0$ is (3, 0, 0).
Hence, the length of perpendicular from point P to the plane containing the triangle OQS is

$$\frac{z-0}{\sqrt{2}}=\frac{3}{\sqrt{2}}$$

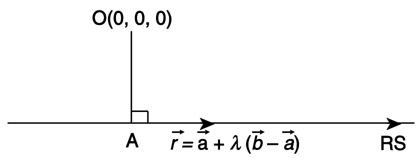
Hence, option (C) is correct.

Now, points R and S, respectively, are R(0, 3, 0) and $S\left(\frac{3}{2},\frac{3}{2},3\right)$.

The equation of line RS is

$$\bar{r}=3\hat{j}+\lambda\left(\frac{3}{2}\hat{i}+\left(\frac{3}{2}-3\right)\hat{j}+3\hat{k}\right)$$

$$=3\hat{j}+\lambda\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)$$



Let point A be the feet of perpendicular from O to line RS:

$$A\left(3\hat{j}+\mu\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)\right)$$

Now, $\overrightarrow{OA}=3\hat{j}+\mu\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)$

That is, $(\vec{b}-\vec{a})=\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)$

Therefore,

$$\overrightarrow{OA}\cdot(\vec{b}-\vec{a})=0$$

$$\left(3\hat{j}+\mu\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)\right).\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)=0$$

$$\frac{-9}{2}+\mu\left(\frac{9}{4}+\frac{9}{4}+9\right)=0$$

$$\frac{-9}{2}+\mu\left(\frac{9}{2}+9\right)=0$$

$$\Rightarrow -\frac{1}{2}+\mu\left(\frac{3}{2}\right)=0$$

$$\Rightarrow \mu=\frac{1}{3}$$

Therefore, $A\left(3\hat{j}+\frac{\hat{i}}{2}-\frac{\hat{j}}{2}+\hat{k}\right)=A\left(\frac{\hat{i}}{2}+\frac{5\hat{j}}{2}+\hat{k}\right)$

and hence the perpendicular distance from point O to the straight line containing the line RS is

$$\left|\overline{OA}\right|=\sqrt{\frac{1}{4}+\frac{25}{4}+\frac{4}{4}}=\sqrt{\frac{30}{4}}=\sqrt{\frac{15}{2}}$$

Hence, option (D) is correct.

8. In a triangle *XYZ*, let *x*, *y*, *z* be the lengths of sides opposite to the angles *X*, *Y*, *Z*, respectively, and $2s=x+y+z$. If $\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}$ and area of incircle of the triangle *XYZ* is $8\pi/3$ then

(A) the area of the triangle *XYZ* is $6\sqrt{6}$.

(B) the radius of circumcircle of the triangle *XYZ* is $\frac{35}{6}\sqrt{6}$.

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}=\frac{4}{35}$.

(D) $\sin^2\left(\frac{X+Y}{2}\right)=\frac{3}{5}$.

Solution

(A), (C), (D) It is given that

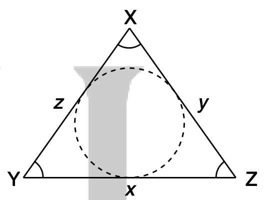
$2s = x + y + z$

Let us consider

$\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}=k$

That is, $s = 4k + x$

$$s = 3k + y$$
$$s = 2k + z$$
$$3s = 9k + (x + y + z) = 9k + 2s$$
$$s = 9k$$
$$x = 9k – 4k = 5k$$
$$y = 6k,\ z = 7k$$



Area of incircle of the triangle XYZ is

$$\pi r^2=\pi\left(\frac{\Delta}{s}\right)^2$$

$=\frac{\pi}{s^2}\Delta^2 \quad =\frac{\pi}{s^2}s(s-x)(s-y)(s-z)$

$=\frac{\pi}{81k^2}\times 9k\times 4k\times 3k\times 2k$

$=\frac{\pi}{9k}\times 24k^3=\frac{\pi}{9}24k^2$

Therefore, $\frac{\pi}{9}24k^2=\frac{8\pi}{3}$

$\Rightarrow k^2=\frac{8\pi}{3}\times\frac{9}{24\pi}=1 \Rightarrow k = 1$

The sides of the triangle is given by $x = 5$, $y = 6$, $z = 7$.

Now, the area of ΔXYZ is

$\sqrt{s(s-x)(s-y)(s-z)} \quad =\sqrt{9\times 4\times 3\times 2}=6\sqrt{6}$

Hence, option (A) is correct.

Now, $R=\frac{xyz}{4\Delta}=\frac{5\times 6\times 7}{4\times 6\sqrt{6}}=\frac{35}{4\sqrt{6}}$

$$\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$$

$$= \sqrt{\left[\frac{(s-y)(s-z)}{yz}\right]\left[\frac{(s-z)(s-z)}{xz}\right]\left[\frac{(s-y)(s-x)}{xy}\right]}$$

$$= \frac{(s-z)(s-y)(s-z)}{xyz}$$

Now, $\frac{(s-z)(s-y)(s-z)}{xyz} = \frac{4\times 3\times 2}{5\times 6\times 7} = \frac{4}{25}$

Hence, option (C) is correct.

Now,

$$\sin^2\left(\frac{X+Y}{2}\right) = \sin^2\left(\frac{\pi - Z}{2}\right) = \cos^2\frac{z}{2} = \frac{s(s-z)}{xy} = \frac{9\times 2}{5\times 6} = \frac{3}{5}$$

Hence, option (D) is correct.

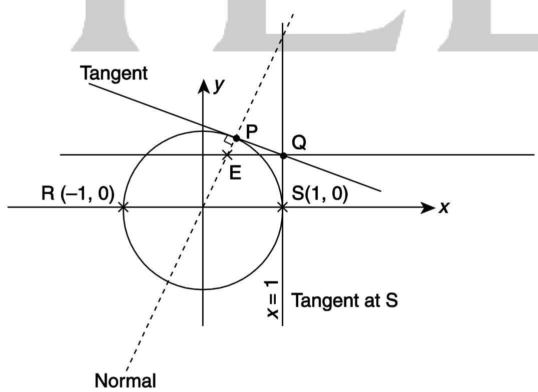
9. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1, 0). Let P be a variable point (other than R and S) on the circle and tangents to the circle as S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E, then the locus of E passes through the point(s)

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$

(C) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4}, \frac{1}{2}\right)$

Solution

(A), (C) The given circle is $x^2 + y^2 = 1$.



The point P on circle is ($\cos\vartheta$, $\sin\vartheta$).

Equation of tangent is $x\cos\vartheta + y\sin\vartheta = 1$ and it meets the tangent at point S.

Now, $\cos\vartheta + y\sin\vartheta = 1 \Rightarrow y = \frac{1-\cos\theta}{\sin\theta}$

Therefore, $\theta\left(1, \frac{1-\cos\theta}{\sin\theta}\right)$

Equation of line through point Q parallel to the line RS is

$$y = \frac{1-\cos\theta}{\sin\theta} \quad (1)$$

The normal to the circle at point P is

$$y = x\tan\vartheta \quad (2)$$

The point of intersection [E(h, k)] of the line [Eq. (1)] and the normal [Eq. (2)] is point is

$\frac{x\sin\theta}{\cos\theta}=\frac{1-\cos\theta}{\sin\theta}\Rightarrow x\sin^2\vartheta = \cos\vartheta - \cos^2\vartheta$

$$\Rightarrow x=\frac{\cos\theta-\cos^2\theta}{\sin^2\theta}=\frac{\cos\theta(1-\cos\theta)}{\sin^2\theta}=h$$

$$\Rightarrow y=\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin^2\theta}\right)(1-\cos\theta)=\frac{1-\cos\theta}{\sin\theta}=k$$

Therefore, $\frac{h}{k}=\frac{\cos\theta}{\sin\theta}$

$\Rightarrow \tan\vartheta = \frac{k}{h}$

$$\Rightarrow \sin\theta=\frac{k}{\sqrt{k^2+h^2}},\ \cos\theta=\frac{h}{\sqrt{h^2+k^2}}$$

That is, $k=\frac{1-\cos\theta}{\sin\theta}$

$$k=\left(1-\frac{h}{\sqrt{h^2+k^2}}\right)\Bigg/\frac{k}{\sqrt{h^2+k^2}}$$

$$k=\frac{\sqrt{h^2+k^2}-h}{k}$$

$$k^2+h=\sqrt{h^2+k^2}$$

$k^4 + h^2 + 2k^2 – h = h^2 + k^2$

$k^2(k^2 + 2h) = k^2$

$k^2 + 2h = 1$

Therefore, the locus of the point E is

$y^2 = 1 – 2x$

which passes through the points $\pm\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$.

Hence, options (A) and (C) satisfy the locus of point E.

10. The circle C_1: $x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then

(A) $Q_2Q_3 = 12$.

(B) $R_2R_3 = 4\sqrt{6}$.

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$.

(D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$.

Solution

(A), (B), (C) The equation of circle C_1 is $x^2 + y^2 = 3$.

The parabola is $x^2 = 2y$.

Point P is obtained as follows:

$y^2 + 2y – 3 = 0$

$y^2 + 3y – y – 3 = 0$

$y(y – 3) – (y + 3) = 0$

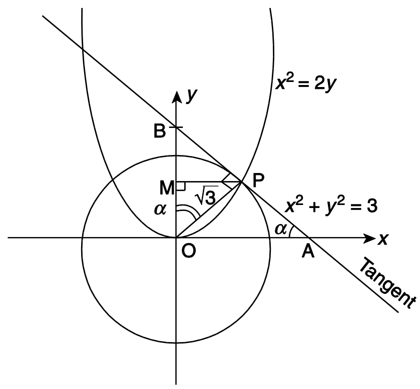
$(y + 3)(y – 1) = 0$

Therefore, $y = −3$ and $y = 1$. Thus, point P is $(\sqrt{2},1)$.

Now, the tangent at point P is $\sqrt{2}x + y = z$.

Point A is $\left(\frac{3}{\sqrt{2}}, 0\right)$.

$$\cos\alpha = \frac{1}{\sqrt{3}} \quad \sin(90^\circ - \alpha) = \frac{Q_2R_2}{Q_2B} = \frac{2\sqrt{3}}{Q_2B}$$



Now, $\angle OBA = 90 - \alpha$; $\sin\alpha = \sqrt{\frac{2}{3}}$;

and $\quad Q_2R_2 = Q_3R_3 = 2\sqrt{3}$

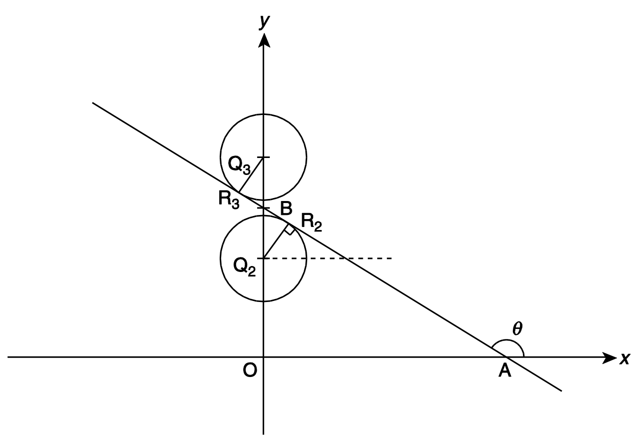
From ΔQ_2BR_2, we get

$$Q_2B = \frac{2\sqrt{3}}{\cos\alpha} = \frac{2\sqrt{3}}{1/\sqrt{3}} = 6$$

From ΔQ_2BR_2, we also get

$$\cos(90^\circ - \alpha) = \frac{BR_2}{Q_2B}$$

That is, $BR_2 = 6(\sin\alpha) = 6\sqrt{\frac{2}{3}}$



From the figure shown here, we see that ΔQ_2BR_2 and Q_3BR_3 are similar triangle. Therefore,

$$Q_2B = Q_3B = 6$$

and $\quad BR_3 = BR_2 = 6\sqrt{\frac{2}{3}}$

Therefore, $\quad Q_2Q_3 = 6 + 6 = 12$

Hence, option (A) is correct.

Now,

$$R_2R_3 = 6\sqrt{\frac{2}{3}} + 6\sqrt{\frac{2}{3}} = 12\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3}\times 144} = \sqrt{2\times 48} = 4\sqrt{6}$$

Hence, option (B) is correct.

Now, the area of OR_2R_3 is

$\frac{1}{2}\times R_2R_3\times OP = \frac{1}{2}\times 4\sqrt{6}\times\sqrt{3} = 2\sqrt{18} = 6\sqrt{2}$

Now, from ΔOPM, we have

$$\sin\alpha = \frac{PM}{OP}$$

$$\sqrt{\frac{2}{3}} = \frac{PM}{\sqrt{3}} \Rightarrow PM = \sqrt{2}$$

The area of ΔPQ_2Q_3 is

$\frac{1}{2}Q_2Q_3(PM) = \frac{1}{2}\times 2\sqrt{2} = 6\sqrt{2}$

Hence, option (C) is correct.

11. Let f: $\mathbb{R}\to\mathbb{R}$, g: $\mathbb{R}\to\mathbb{R}$ and h: $\mathbb{R}\to\mathbb{R}$ be differentiable functions such that $f(x) = x^2 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x\in\mathbb{R}$. Then

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$

(C) $h(0) = 16$ (D) $h(g(3)) = 36$

Solution

(B), (C)

(A) It is given that f: $\mathbb{R}\to\mathbb{R}$, g: $\mathbb{R}\to\mathbb{R}$ and h: $\mathbb{R}\to\mathbb{R}$ are differentiable functions.

Now, $f(x) = x^3 + 3x + 2$

Differentiating w.r.t. to x, we get

$f'(x) = 3x^2 + 3$

Also, $g(f(x)) = x$

Now, $g'(f(x))\cdot f'(x) = 1$

$$f(x) = 2 \Rightarrow x^3 + 3x + 2 = 2$$

$$\Rightarrow x^3 + 3x = 0 \Rightarrow x(x^2 + 3) = 0$$

$$\Rightarrow x = 0$$

Now, $g'(f(0)) = \frac{1}{f'(0)} \Rightarrow g'(2) = \frac{1}{3}$

Hence, option (A) is incorrect.

(B) For all $x\in\mathbb{R}$:

$h(g(g(f(x))) = x$

$h(g(g(x))) = x$

Now, $x\to f(x) \Rightarrow h(g(g(f(x))) = f(x)$

$\Rightarrow h(g(x)) = f(x)$

$\Rightarrow h'(g(x))\cdot g'(x) = f'(x) = 3x^2 + 3$ (1)

For all $x\in\mathbb{R}$:

$g(f(x)) = x$

Now, $x = 1 \Rightarrow g(f(1)) = 1 \Rightarrow g(6) = 1$ [$\because f(1) = 6$]

Substituting $x = 6$ in Eq. (1), we get

$h'(g(6))\cdot g'(6) = 3(6^2) + 3 = 111$

Therefore, $h'(1) = \frac{111}{g'(6)} \left(g'(6) = \frac{1}{f'(1)} \right)$

That is, $h'(1) = 111 \cdot f'(x) = 111 \times (3+3) = 666$

Hence, option (B) is correct.

(C) $h(g(g(x))) = x.$

For $g(g(x)) = 0$, we have

$g(x) = g^{-1}(0) = 2$

$\Rightarrow x = g^{-1}(2) = f(2) = 16$

$\Rightarrow h(0) = 16$

Hence, option (C) is correct.

(D) Here, $g(g(x)) = g(3)$ which implies that

$g(x) = 3 \Rightarrow x = g^{-1}(3) = f(3) = 38$

Hence, option (D) is incorrect.

12. Let $f:(0,\infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1.$ Then

(A) $\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$

(B) $\lim_{x\to 0^+} xf\left(\frac{1}{x}\right) = 2$

(C) $\lim_{x\to 0^+} x^2 f'(x) = 0$

(D) $|f(x)| \le 2$ for all $x \in (0,2)$

Solution

(A) It is given that

$f:(0,\infty) \to \mathbb{R}, \quad f'(x) = 2 - \frac{f(x)}{x}$

Now, the linear differential equation is

$f'(x) + \frac{f(x)}{x} = 2$

The integrating factor is

$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Therefore, $\int d(xf(x)) = \int 2x\,dx + c$

$xf(x) = x^2 + c \Rightarrow f(x) = \left(x + \frac{c}{x}\right) \quad \forall x \in (0,\infty)$

Now, $f(1) \neq 1 \Rightarrow 1 \neq 1 + c_1 \Rightarrow c \neq 0$

and $f'(x) = 1 - \frac{c}{x^2} \Rightarrow \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+}\left(1 - \frac{c}{x^2}\right) = 1$

$\Rightarrow \lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x\to 0^+}(1 - cx^2) = 1$

Hence, option (A) is correct.

Now, $\lim_{x\to 0^+} x\, f\left(\frac{1}{x}\right) = \lim_{x\to 0^+} x\left(\frac{1}{x} + a\right) = \lim_{x\to 0^+}(1 + (x^2)) = 1$

Hence, option (B) is incorrect.

Now, $\lim\limits_{x\to0^+} x^2 f'(x) = \lim\limits_{x\to0^+} x^2\left(1-\frac{c}{x^2}\right) = \lim\limits_{x\to0^+}(x^2-c) = -c \ne 0$

Hence, option (C) is incorrect.

We cannot say anything about $|f(x)| \le 2\ \forall x \in (0,2)$ because we do not know the value of c.

Hence, option (D) is incorrect.

13. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R}$. Suppose $Q = [q_{ij}]$ is a matrix such that PQ = kI, where $k \in \mathbb{R}, k \ne 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and det(Q) $= \frac{k^2}{2}$, then

(A) $\alpha = 0$, $k = 8$ (B) $4\alpha - k + 8 = 0$

(C) det(Padj(Q)) = 2^9 (D) det(Qadj(P)) = 2^{13}

Solution

(B), (C) It is given that $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$ and $Q = [qij]$.

Now, $C_{32} = -\begin{vmatrix} 3 & -2 \\ 2 & \alpha \end{vmatrix} = -(3\alpha + 4)$

Here, $|P| = 12\alpha + 20$

$PQ = kI$

$Q = kIP^{-1}$

Now, $|P|\,|Q| = k^3$

Therefore, $(12\alpha + 20)\frac{k^2}{2} = k^3$

$6\alpha + 10 = k$ (1)

That is, $|P| = 2k$

$Q = kI\frac{(\text{adj } P)}{|P|} = \frac{(\text{adj } P)}{2}$

Now, $q_{23} = \frac{1}{2}C_{32} = \frac{-(3\alpha+4)}{2} = \frac{-k}{8}$

$\Rightarrow 12\alpha + 16 = k$ (2)

Solving Eqs. (1) and (2), we get

$-4 = -k \Rightarrow k = 4$

Using the value of k in Eq. (1), we get

$6\alpha + 10 = 4 \Rightarrow \alpha = -1$

That is, $\alpha = -1$ and $k = 4$.

Hence, option (A) is incorrect.

The values of $\alpha = -1$ and $k = 4$ satisfy the equation given in option (B).

Hence, option (B) is correct.

Now, $(\det Q) = \frac{k^2}{2} = 8$

Therefore,

$$\det(P\text{adj}Q) = (\det P)\det(\text{adj}Q)$$

$$= (2 \times 4)(\det Q)^2 = 8 \times 2^2 = 2^3 \times 2^6 = 2^9$$

Hence, option (C) is correct and option (D) is incorrect.

Integer Answer Type

This section contains 5 questions. The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9 (both inclusive).

14. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)^{51}C_3$ for some positive integer n. Then the value of n is ________.

Solution

It is given that the coefficient of x^2 in $(1 + x)^2 + (1 + x)^3 + \ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$.

Now, ${}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^{49}C_2 + m^2\ {}^{50}C_2 = (3n + 1)\ {}^{51}C_3$

$\Rightarrow {}^3C_3 + {}^3C_2 + {}^4C_2 + \ldots + {}^{49}C_2 + m^2\ {}^{50}C_2 = (3n + 1)\ {}^{51}C_3$

($\because$ $nCr + nCr_{-1} = n^{+1}Cr$)

$\Rightarrow {}^4C_3 + {}^4C_2 + \ldots + {}^{49}C_2 + m^2\ {}^{50}C_2 = (3n + 1)\ {}^{51}C_3$

$\Rightarrow {}^{49}C_3 + {}^{49}C_2 + m^2\ {}^{50}C_2 = (3n + 1)\ {}^{51}C_3$

$\Rightarrow {}^{50}C_3 + m^2\ {}^{50}C_2 = (3n + 1)\ {}^{51}C_3$

$\Rightarrow {}^{50}C_3 + {}^{50}C_2 + m^2\ {}^{50}C_2 = (3n + 1)\ {}^{51}C_3 + {}^{50}C_2$

$\Rightarrow {}^{51}C_3 + m^2\ {}^{50}C_2 = 3n\ {}^{51}C_3 + {}^{51}C_3 + {}^{50}C_2$

$\Rightarrow {}^{-50}C_2 + m^2\ {}^{50}C_2 = 3n\ {}^{51}C_3$

$\Rightarrow {}^{50}C_2\ (m^2 - 1) = 3n \cdot \frac{51}{3}\ {}^{50}C_2 \left({}^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1}\right)$

$\Rightarrow 2(m^2 - 1) = 51$

Therefore, $m^2 = 51n + 1$

$$\Rightarrow n = \frac{m^2 - 1}{51}$$

$$\Rightarrow m = 16 \Rightarrow n = 5\ (m, n \in \mathrm{I}^+)$$

15. The total number of distinct $x \in \mathbb{R}$ for which $\begin{vmatrix} x & x^2 & 1+x^2 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is __________.

Solution It is given that

$$\begin{vmatrix} x & x^{-2} & 1+x^3 \\ 2x & (2x)^2 & 1+(2x)^3 \\ 3x & (3x)^2 & 1+(3x)^3 \end{vmatrix} = 10$$

That is, $\begin{vmatrix} x & x^2 & 1 \\ 2x & 4x^2 & 1 \\ 3x & 9x^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 2x & 4x^2 & 8x^3 \\ 3x & 9x^2 & 27x^3 \end{vmatrix} = 10$

$$x^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 1 \\ 3 & 3^2 & 1 \end{vmatrix} + x^6 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{vmatrix} = 10$$

$$x^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 1 \\ 3 & 3^2 & 1 \end{vmatrix} + 6x^6 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{vmatrix} = 10$$

$$x^3(1+6x^3)\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 3 & 9\end{vmatrix}=10$$

- $C2 \rightarrow C2 - C1$ ανδ $C3 \rightarrow C3 - C1$:

$$x^3(1+6x^3)\begin{vmatrix}1 & 0 & 0\\ 1 & 1 & 3\\ 1 & 2 & 8\end{vmatrix}=10$$

$x^3(1 + 6x^3)\ 1(8 - 6) = 10$
$x^3(1 + 6x^3) = 5$
$6(x^3)^2 + x^3 - 5 = 6$
$6(x^3)^2 + 6x^3 - 5x^3 - 5 = 0$
$6x^3(x^3 + 1) - 5(x^3 + 1) = 0$
$(6x^3 - 5)\ (x^2 - x + 1)\ (x + 1) = 0$

Therefore, $x = -1$ and $(5/6)^{1/3}$.

Hence, there are two district values for x. Thus, 2 is the answer.

16. Let $z=\frac{-1+\sqrt{3}i}{2}$, where $i=\sqrt{-1}$, and r, $s \in \{1,2,3\}$. Let $P=\begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is ________.

Solution

It is given that $z=\frac{-1}{2}+i\frac{\sqrt{3}}{2}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=e^{i2\pi/3}=\omega$ (cube root of unity), where r, $s \in \{1, 2, 3\}$.

It is also given that $P=\begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

Since $P^2 = -I$, we have

$$P^2=\begin{bmatrix}(-z)^r & Z^{2s}\\ Z^{2s} & Zr\end{bmatrix}\begin{bmatrix}(Z)^r & Z^{2s}\\ Z^{2s} & Z^r\end{bmatrix}=-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}(-Z)^{2r}+Z^{4s} & (-Z)^r\cdot Z^{2s}+Z^rZ^{2s}\\ (-Z)^rZ^{2s}+Z^r\cdot Z^{2s} & Z^{4s}+(Z)^{2r}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

That is, we have $(-Z)^2r + Z^4s = -1$ and $Z^4s + Z^2r = -1$

and $((-Z)r + Zr)Z^2s = 0$ and $Z^2r + Z^4s = -1$

That is, $((-\omega)r + (\omega)r)\cdot\omega^2s = 0$

Now, $\omega^2s \neq 0$; therefore,

$(-\omega)r + (\omega)r = 0$

where r is the odd number and hence $r = 1, 3$.

When $r = 1$: $(-\omega)^2 + \omega^4s = -1 \Rightarrow \omega^4s = -1 - \omega^2 = +\omega$.

Now, s can be 1 (since $s \neq 3$).

That is, $(r, s) = (1, 1)$, that is, the total number of ordered pair (r, s) is one (single) for which $P^2 = -I$.

Hence, the answer is 1.

17. The total number of distinct $x\in[0,1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt=2x-1$ is ________.

Solution

We have $x \in [0,1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x-1.$

Let $F(x) = \int_0^x \frac{t^2}{1+t^4}dt$

and $f(x) = 2x - 1$

Now, if $F'(x) = \frac{x^2}{1+x^4} > 0$, it means that $F(x)$ is an increasing function.

$F(0) = 0$

$$F(x) = \frac{1}{2}\int_0^x \frac{(t^2+1)+(t^2-1)}{1+t^4}dt$$

$$= \frac{1}{2}\int_0^x \left(\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2} + \frac{\left(1-\frac{1}{t^2}\right)}{\left(t+\frac{1}{t}\right)^2-2} \right) dt$$

$$= \frac{1}{2}\int_0^x \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+2} + \frac{1}{2}\int_0^x \frac{\left(1-\frac{1}{t^2}\right)}{\left(t+\frac{1}{t}\right)^2-2}dt$$

$$= \frac{1}{2\sqrt{2}}\left\{ \tan^{-1}\left(t-\frac{1}{t}\right) + \ln\left(\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}} \right) \right\}\Bigg|_0^x$$

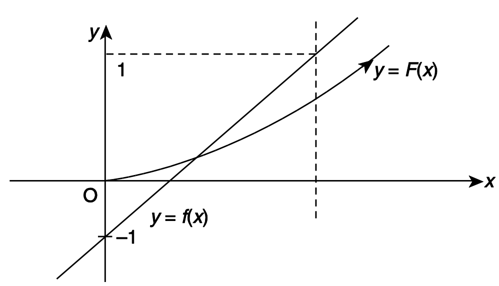
$$= \frac{1}{2\sqrt{2}}\left\{ \tan^{-1}\left(\frac{t^2-1}{t}\right) + \ln\left(\frac{t^2+1-\sqrt{2t}}{t^2+1+\sqrt{2t}} \right) \right\}\Bigg|_0^x$$

$$= \frac{1}{2\sqrt{2}}\left\{ \tan^{-1}\left(\frac{x^2-1}{x}\right) + \ln\left(\frac{x^2+1-\sqrt{2t}}{x^2+1+\sqrt{2t}} \right) - \left(-\frac{\pi}{2}+0\right) \right\}$$

Therefore,

$$F(x) = \frac{1}{2\sqrt{2}}\frac{\pi}{2} + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right)$$

$$F(1) = \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}(0) + \frac{1}{2\sqrt{2}}\ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) < 1$$



Now, $f(x) = 2x - 1 \Rightarrow f(0) = -1$

and $f(1) = 2 - 1 = 1$

Therefore, the total number of distinct values of $x \in [0.\ 1]$ is only one.

Hence, the answer is 1.

18. Let $\alpha, \beta \in \mathbb{R}$ be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{ax-\sin x} = 1.$ Then, $6(\alpha + \beta)$ equals ________.

Solution

It is given that $\alpha, \beta \in \mathbb{R}$ such that $\lim_{x\to 0}\frac{x^2 \sin \beta x}{\alpha x - \sin x} = 1.$ Therefore,

$$\lim_{x\to 0}\frac{x^2 \sin \beta x}{\alpha x - \sin x} = 1$$

$$\Rightarrow \lim_{x\to 0}\frac{x^2\left(\frac{\sin \beta x}{\beta x}\right)\beta x}{\alpha x - \sin x} = 1$$

$$\Rightarrow \beta \lim_{x\to 0}\frac{x^3}{\alpha x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)} = 1$$

$$\Rightarrow \beta \lim_{x\to 0}\frac{x^3}{x(\alpha - 1) + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots} = 1$$

For finite limit $\alpha = 1$,

$3! \times \beta = 1 \Rightarrow \beta = \frac{1}{6}$

Then, $6(\alpha + \beta) = 6\left(1 + \frac{1}{6}\right) = 6 + 1 = 7$