JEE ADVANCED | 2016
PAPER 2

MATHEMATICS

MATHEMATICS

Single Option Correct Type

This section contains 5 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

1. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [qij]$ is a matrix such that $P^{50} - Q = I$, then $\dfrac{q_{31} + q_{32}}{q_{21}}$ equals

(A) 52 (B) 103
(C) 201 (D) 205

Solution

(B)

It is given that

$P^{50} - Q = I$

That is, $Q = P^{50} - I = [qij]_{3\times3}$

The given matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & a & 1 \end{bmatrix}$$

Let us consider $a = 4$; therefore,

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & a & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 3a^2 & 2a & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 3a^2 & 2a & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3a & 1 & 0 \\ 6a^2 & 3a & 1 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 1 & 0 & 0 \\ 4a & 1 & 0 \\ 10a^2 & 4a & 1 \end{bmatrix}$$

$$P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 50a & 1 & 0 \\ T_{50} & 50a & 1 \end{bmatrix}$$

Therefore, using difference method, we get

$$S_{50} = a^2 + 3a^2 + 6a^2 + 10a^2 + \cdots + T_{50}$$
$$S_{50} = \quad a^2 + 3a^2 + 6a^2 + \cdots + T_{59} + T_{50}$$

$$T_{50} = a^2(1 + 2 + 3 + \cdots + 50) = \frac{a^2(50)(51)}{2} = a^2(25)(51)$$

Therefore, $I^{50} - I = \begin{bmatrix} 0 & 0 & 0 \\ 50a & 0 & 0 \\ T_{50} & 50a & 0 \end{bmatrix} = [qij]_{3\times 3}$

Therefore, we get the following values:

$$q_{31} = T_{50} = 25(51)a^2,$$
$$q_{32} = 50a$$
$$q_{21} = 50a$$

Therefore,

$$\frac{q_{31} + q_{32}}{q_{21}} = \frac{(25)(51)(a^2)}{50a} + 1 = 1 + \frac{(25)(51)(16)}{(50)(4)} = 103$$

2. The area of the region

$\{(x, y) \in \mathbb{R}^2 : y \geq \sqrt{|x+3|}, 5y \leq x \ + 9 \leq 15\}$

is equal to

(A) $\frac{1}{6}$ (B) $\frac{4}{3}$

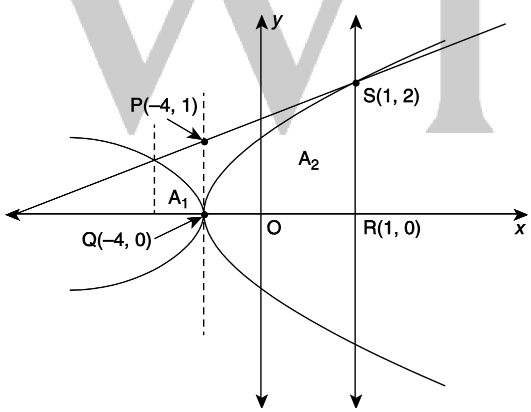
(C) $\frac{3}{2}$ (D) $\frac{5}{3}$

Solution

(C)

From the figure shown here, the following condition is obvious:

Area of the trapezium PQRS − $(A_1 + A_2)$



Now, the area of the trapezium PQRS is

$$\frac{1}{2} \times (1+2) \times 5 = \frac{15}{2}$$

The area A_1 is obtained as

$$\int_{-4}^{-3} \sqrt{-(x+3)}\, dx = \frac{2}{3}$$

The area A_2 is calculated as

$$\int_{-3}^{1} \sqrt{(x+3)^{1/2}}\, dx = \frac{16}{3}$$

Therefore, from Eq. (1), we calculate the area of the given as follows:

$$\frac{15}{2}-\left(\frac{2}{3}+\frac{16}{3}\right)=\frac{3}{2}\text{sq.unit}$$

3. The value of $\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to

(A) $3-\sqrt{3}$ (B) $2(3-\sqrt{3})$

(C) $2(\sqrt{3}-1)$ (D) $2(2+\sqrt{3})$

Solution

(C)

It is given that

$$\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$$

Let $\alpha=\frac{\pi}{4}$ and $\beta=\frac{\pi}{6}$. Therefore,

$$\sum_{k=1}^{13}\frac{1}{\sin(\alpha+k\beta)\sin(\alpha+(k-1)\beta)}$$

$$=\frac{1}{\sin\beta}\sum_{k=1}^{13}\frac{\sin((\alpha+k+\beta)-(\alpha+(k-1)\beta))}{\sin(\alpha+k\beta)\sin(\alpha+(k-1)\beta)}$$

$$=\frac{1}{\sin\beta}\sum_{k=1}^{13}(\cot(\alpha+(k-1)\beta)-\cot(\alpha+k\beta))$$

$$=\frac{1}{\sin\beta}\{[\cot(\alpha)-\cot(\alpha+\beta)]+[\cot(\alpha+\beta)-\cot(\alpha+2\beta)]+\cdots$$

$$\cdots+[\cot(\alpha+12\beta)-\cot(\alpha+13\beta)]\}$$

$$=\frac{1}{\sin\beta}(\cot\alpha-\cot(\alpha+13\beta))$$

$$=\frac{1}{\sin(\pi/6)}\left(\cot\frac{\pi}{4}-\cot\left(\frac{\pi}{4}+\frac{13\pi}{6}\right)\right)$$

$$=2(1-2+\sqrt{3})=2(\sqrt{3}-1)$$

4. Let $b_i > 1$ for $I = 1, 2, \ldots, 101$. Suppose $\log_e b_1, \log_e b_2, \ldots, \log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, \ldots, a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \ldots + b_{51}$ and $s = a_1 + a_2 + \ldots + a_{51}$, then

(A) $s > t$ and $a_{101} > b_{101}$

(B) $s > t$ and $a_{101} < b_{101}$

(C) $s < t$ and $a_{101} > b_{101}$

(D) $s < t$ and $a_{101} < b_{101}$

Solution

(B)

Let $bi > 1$ and $i = 1, 2, \ldots, 101$. Therefore,

$\log_e b_1, \log_e b_{21}, \ldots, \log_e b_{101}$ in A.P.. (common difference = $\log_e 2$)
$b_1, b_{21}, \ldots, b_{101}$ in G.P. (common ratio = 2)
We see that $a_1, a_{21}, \ldots, a_{101}$ are in A.P. Therefore,

$$a_1 = b_1$$

$$a_{51} = b_{51}$$

That is, $b_{51} = b_1(2^{50}) \Rightarrow \frac{b_{51}}{b_1} = 2^{50}$

Therefore, $a_{51} = a_1 + 50d$

where d is the common difference of the second given A.P.

Therefore, $b_{51} = b_1 + 50d \Rightarrow 50d = b_{51} - b_1 \Rightarrow d = \frac{(b_{51} - b_1)}{50}$

Now, $a_{101} = a_1 + 100d = a_1 + 100\frac{(b_{51} - b_1)}{50}$

$= (2b_{51} - b_1) - [2(2^{50})b_1 - b_1]$

$= b_1(2^{51} - 1)$

and $b_{101} = b_1 2^{100}$

That is, $b_{101} > a_{101}$

Hence, $t = b_1 + b_2 + \cdots + b_{51} = \frac{b_1(2^{50} - 1)}{(2-1)} = b_1(2^{50} - 1)$

and $s = \left(\frac{a_1 + a_{51}}{2}\right)51 = \frac{51}{2}(b_1 + b_{51})$

$= \frac{51}{2}(b_1 + b_1 2^{50}) = \frac{51 b_1}{2}(2^{50} + 1)$

Therefore, it is obvious that $s > t$.

Hence, option (B) is correct.

5. The value of $\int_{-(\pi/2)}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx$ is equal to

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$

(C) $\pi^2 - e^{-(\pi/2)}$ (D) $\pi^2 + e^{(\pi/2)}$

Solution

(A)

The given integral is

$$I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx$$

Using the integral property, we get

$$I = \int_0^{\pi/2} \left(\frac{x^2 \cos x}{1+e^x} + \frac{x^2 \cos x}{1+e^{-x}}\right) dx$$

$$I = \int_0^{\pi/2} x^2 \cos x \, dx$$

That is, $\int x^2 \cos x \, dx = x^2 \sin x - 2\int x \sin x \, dx$

$= x^2 \sin x - 2\{-x \cos x + \int \cos x \, dx\}$

$= x^2 \sin x + 2x \cos x - 2\sin x$

Therefore,

$$I = x^2 \sin x + 2x \cos x - 2\sin x \Big|_0^{\pi/2}$$

$$= \left(\frac{\pi^2}{4} + 0 - 2\right) - (0) = \frac{\pi^2}{4} - 2$$

6. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then, the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is

(A) $x + y - 3z = 0$ (B) $3x + z = 0$

(C) $x - 4y + 7z = 0$ (D) $2x - y = 0$

Solution

(C)

Let P(x_1, y_1, z_1) image of Q(3, 1, 7) w.r.t. the plane $x - y + z = 3$.

Let R be the point on plane which is midpoint of the line joining P and Q.

The equation of the line PQ is

$$\frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-7}{1} = \lambda$$

Therefore, $(x, y, z) = (3 + \lambda, 1 - \lambda, 7 + \lambda)$ lies on plane.

$3 + \lambda - 1 + \lambda + 7 + \lambda = 3$

$3\lambda + 6 = 0 \Rightarrow \lambda = -2$

The point R is (3 – 2, 3, 7 – 2) = (1, 3, 5).

Now, $\frac{x_1 + 3}{2} = 1 \Rightarrow x_1 = -1$

$\frac{y_1 + 1}{2} = 3 \Rightarrow y_1 = 5$

$\frac{z_1 + 7}{2} = 5 \Rightarrow z_1 = +3$

That is, the point P is P(−1, 5, +3).

Now, the equation of the plane passing through P is

$a(x + 1) + b(y - 5) + c(z - 3) = 0$

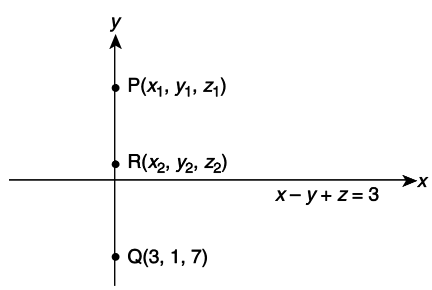
This plane contains the line:

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

That is, $a(0 + 1) + b(0 - 5) + c(0 - 3) = 0 \Rightarrow a = 5b + 3c$

$$a + 2b + c = 0 \Rightarrow 7b + 4c = 0 \Rightarrow b = \frac{-4c}{7}$$

$$a = -\frac{20c}{7} + 3c = \frac{c}{7}$$



Now, the equation of the plane is obtained as follows:

$$\frac{c}{7}(x+1) - \frac{4c}{7}(y-5) + c(z-3) = 0$$

$$(x + 1) - 4(y - 5) + 7(z - 3) = 0$$

$$x + 1 - 4y + 20 + 7z - 21 = 0$$

$$x - 4y + 7z = 0$$

One or More Than One Option Correct Type

This section contains 8 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONE OR MORE** is(are) correct.

7. Let $a, b \in \mathbb{R}$ and f: $\mathbb{R} \to \mathbb{R}$ be defined by

$f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$.

Then f is

(A) differentiable at $x = 0$ if $a = 0$ and $b = 1$.

(B) differentiable at $x = 1$ if $a = 1$ and $b = 0$.

(C) NOT differentiable at $x = 0$ if $a = 1$ and $b = 0$.

(D) NOT differentiable at $x = 1$ if $a = 1$ and $b = 1$.

Solution

(A), (B)

The given function is

$f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$

which is an even function.

$f(x) = a\cos(x^3 - x) + bx\sin(x^3 + x)$

For a function to be differentiable at $x = 0$, the function must be continuous.

$f(0) = a\cos(0 - 0) + b(0)\sin(0) = a$

Therefore,

$$f(0^+) = \lim_{h\to 0}[a\cos(h^3 - h) + bh\sin(h^3 + h)] = 0$$

$$f(0^-) = \lim_{h\to 0}[a(\cos(-h^3 + h) + b(-h)\sin(-h^3 - h)]$$

$$= \lim_{h\to 0}[a\cos(h^3 - h) + bh\sin(h^3 + h)] = 0$$

which is continuous at $x = 0$; hence, $f(x)$ is differentiable for all values of a and b.

Therefore,

$f(1) = a\cos(1 - 1) + 1\sin(1 + 1) = a + b\sin 2$

$$f(1^+) = \operatorname*{lt}_{h\to 0} a\cos(1+h)^3 - (1+h) + b(1+h)\sin(1+h)^3 + (1+h)$$

$$= f(1)$$

$$f(1^-) = \operatorname*{lt}_{h\to 0} a\cos(1-h)^3 - (1-h) + b(1-h)\sin(1-h)^3 + (1-h)$$

$$= f(1)$$

Thus, $f(x)$ is continuous and we can also see that f is differentiable at $x = 0$ and $x = 1$.

Hence, options (A) and (B) are correct.

8. Let $f(x) = \lim_{n\to\infty}\left(\dfrac{n^n(x+n)\left(x+\frac{n}{2}\right)\cdots\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right)\cdots\left(x^2+\frac{n^2}{n^2}\right)}\right)^{x/n}$, for all $x > 0$. Then

(A) $f\left(\frac{1}{2}\right) \geq f(1)$ (B) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$

(C) $f'(2) \leq 0$ (D) $\dfrac{f'(3)}{f(3)} \geq \dfrac{f'(2)}{f(2)}$

Solution

(B), (C)

The given function is

$$f(x)=\lim_{n\to\infty}\left(\frac{n^n(x+n)(x+n/2)\cdots(x+n/n)}{n!(x^2+n^2)(x^2+n^2/4)\cdots(x^2+n^2/n^2)}\right)^{x/n}\quad \forall x>0$$

That is,

$$f(x)=\lim_{n\to\infty}\left(\frac{n^n n^n\left(1+\frac{x}{n}\right)\left(1+\frac{2x}{n}\right)\cdots\left(1+\frac{nx}{n}\right)}{n!n!\left(1+\frac{x^2}{n^2}\right)\left(1+\frac{4x^2}{n^2}\right)\cdots\left(1+\frac{n^2x^2}{n^2}\right)}\times\frac{(n!)^2}{(n^2)^n}\right)^{x/n}$$

$$f(x)=\lim_{n\to\infty}\left(\frac{\left(1+\frac{x}{n}\right)\left(1+\frac{2x}{n}\right)\cdots\left(1+\frac{nx}{n}\right)}{\left(1+\frac{x^2}{n^2}\right)\left(1+\frac{4x^2}{n^2}\right)\cdots\left(1+\frac{n^2x^2}{n^2}\right)}\right)$$

$$=\lim_{n\to\infty}\left(\prod_{r=1}^{n}\left(\frac{1+\frac{rx}{n}}{1+\frac{r^2x^2}{n^2}}\right)\right)^{x/n}$$

Therefore, $\ln f(x)=x\lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{n}\ln\left(\frac{1+\frac{rx}{n}}{1+\frac{r^2x^2}{n^2}}\right)\right)$

$$\ln f(x)=x\int_0^1\ln\left(\frac{1+xy}{1+x^2y^2}\right)dy$$

Substituting $xy = t \Rightarrow dy = \frac{dt}{x}$. Therefore,

$$\ln f(x)=\frac{x}{x}\int_0^x\ln\left(\frac{1+t}{1+t^2}\right)dt=\int_0^x\ln\left(\frac{1+t}{1+t^2}\right)dt$$

Applying Newton–Lebniz rule, we get

$$\frac{f'(x)}{f(x)}=\ln\left(\frac{1+x}{1+x^2}\right)\Rightarrow f'(x)=f(x)\ln\left(\frac{1+x}{1+x^2}\right)$$

It is obvious that $f(x)>0\ \forall x>0.$

For $x = 2$: $\quad\frac{f'(2)}{f(2)}=\ln\left(\frac{3}{5}\right)<0\ \Rightarrow f'(2)\le 0$

Hence, option (C) is correct.

That is, $f'(x)\ge 0\ \forall x\in[0,1]\Rightarrow f(x)$ is an increasing function:

That is, $f\left(\frac{1}{3}\right)\le f\left(\frac{x}{3}\right)$

Hence, option (B) is correct.

9. Let $f:\mathbb{R}\to(0,\infty)$ and g: $\mathbb{R}\to\mathbb{R}$ be twice differentiable function such that f'' and g'' are continuous functions on $\mathbb{R}$. Suppose $f'(2)=g(2)=0,\ f''(2)\ne g'(2)\ne 0.$ If $\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)}=1,$ then

(A) f has a local minimum at $x = 2$.

(B) f has a local maximum at $x = 2$.

(C) $f''(2)>f(2)$.

(D) $f(x)-f''(x)=0$ for at least one $x\in\mathbb{R}$.

Solution

(A), (D)

Let $f:\mathbb{R}\to(0,\infty)$ and g: $\mathbb{R}\to\mathbb{R}$.

$f(x) > 0 \ \forall x\in\mathbb{R}$

It is given that $f'(2)=0$, $g(2) = 0$, $f''(2)\neq 0$ and $g'(2)\neq 0$.

It is also given that

$$\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)}=1 \qquad \left(\frac{0}{0}\right)$$

Applying L'Hospital rule, we get

$$\lim_{x\to 2}\frac{f'(x)g(x)+g'(x)f(x)}{f''(x)g'(x)+f'(x)g''(x)}=1$$

For finite limit, we get

$$\frac{f'(2)g(2)+g'(2)f(2)}{f''(2)g'(2)+f'(2)g''(2)}=1$$

$$\frac{g'(2)f(2)}{f''(2)g'(2)}=1$$

$$\frac{f(2)}{f''(2)}=1 \Rightarrow f''(2)=f(2)>0 \text{ and } f'(2)=0$$

which means that $f(x)$ has local minima at $x = 2$.

Hence, option (A) is correct.

$f(2)-f''(2)=0$

Therefore, we can say that $f(x)-f''(x)=0$ has at least one solution in $x\in\mathbb{R}$.

Hence, option (D) is correct.

10. Let $\hat{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$ be a unit vector in $\mathbb{R}^2$ and $\hat{\omega}=\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$. Given that there exists a vector $\vec{v}$ in $\mathbb{R}^3$ such that $|\hat{u}\times\vec{v}|=1$ and $\hat{\omega}\cdot(\hat{u}\times\vec{v})=1$. Which of the following statement(s) is(are) correct?

(A) There is exactly one choice for such $\vec{v}$.

(B) There are infinitely many choice for such $\vec{v}$.

(C) If $\hat{u}$ lies in the xy-plane, then $|u_1| = |u_2|$.

(D) if $\hat{u}$ lies in the xz-plane, then $2|u_1| = |u_3|$.

Solution

(B), (C)

We have

$\hat{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$

That is, $|\hat{u}|=1=\sqrt{u_1^2+u_2^2+u_3^2}$

$\Rightarrow u_1^2+u_2^2+u_3^2=1$

Also, it is given that

$\hat{\omega}=\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$

That is, $|\hat{\omega}|=1$

Now, $|\hat{u}\times\vec{v}|=1$

That is, $|\hat{u}||\vec{v}|\sin\theta=1 \Rightarrow |\vec{v}|=\frac{1}{\sin\theta}$

which shows that there are infinitely many possible values exist for $\vec{v}$ (here ϑ is angle between the vectors $\vec{v}$ and $\hat{u}$.

Hence, option (B) is correct.

Now, $\hat{\omega} \cdot (\hat{u} \times \vec{v}) = 1$

$|\hat{\omega} \cdot (\hat{u} \times \vec{v})| = 1$

That is, $|\hat{\omega}||\hat{u} \times \vec{v}| \cos\alpha = 1$

where α is the angle between $\hat{\omega}$ and $\hat{u} \times \vec{v}$.

Therefore, $(1)(1)\cos\alpha = 1$

$\Rightarrow \alpha = 0$

which means that $\hat{\omega}$ and $\hat{u} \times \vec{v}$ are parallel vector or $\hat{\omega}$ is perpendicular vector to $\hat{u}$ and $\vec{v}$.

$$\hat{u} \cdot \hat{\omega} = 0$$

$$(u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) \cdot \left(\frac{(\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{6}} \right) = 0$$

$$u_1 + u_2 + 2u_3 = 0$$

If $\hat{u}$ lies in xy-plane then $u_3 = 0$. Therefore,

$$u_1 + u_2 = 0 \Rightarrow u_1 = -u_2 \Rightarrow |u_1| = |u_2|$$

Hence, option (C) is correct.

11. Let P be the point on the parabola $y^2 = 4x$, which is at the shortest distance from the centre S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then

(A) $SP = 2\sqrt{3}$.

(B) $SQ : QP = (\sqrt{5} + 1) : 2$.

(C) the x-intercept of the normal to the parabola at P is 6.

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$.

Solution

(A), (C), (D)

The given parabola is

$y^2 = 4x$

The point on the parabola is $P(am^2, -2am)$.

Therefore, $4a = 4 \Rightarrow a = 1$

Hence, the point P becomes $P(m_1^2 - 2m)$.

The equation of normal to parabola at point $P(m_1^2 - 2m)$ is

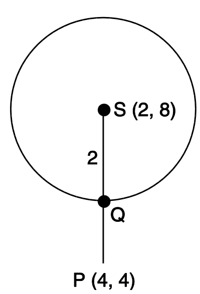
$y = mx - 2m - m^3$

For the shortest distance from circle $x^2 + y^2 - 4x - 16y + 64 = 0$, this normal is also normal to the given circle; therefore, it passes through the centre of the circle (2, 8).

$$8 = 2m - 2m - m^3$$

$$m^3 = -8 \Rightarrow m = -2$$

Therefore, point P is (4, 4).



The radius of the circle is

$\sqrt{4 + 64 - 64} = 2$

Therefore, the segment SP is given by

$$SP=\sqrt{(4-2)^2+(4-9)^2}=\sqrt{20}=2\sqrt{5}$$

Hence, option (A) is correct.

Now,

$$\frac{SQ}{QP}=\frac{2}{SP-SQ}=\frac{2}{2\sqrt{5}-2}=\frac{1}{\sqrt{5}-1}\times\frac{\sqrt{5}+1}{\sqrt{5}+1}=\frac{\sqrt{5}+1}{4}$$

The normal of the parabola at point P is given by

$y=-2x+4+8 \Rightarrow 2x+y=12$

Therefore, the x intercept is 6.

Hence, option (C) is correct.

Therefore, the slope of the tangent to the circle at point Q is $+1/2$.

Hence, option (D) is correct.

12. Let $a,b\in\mathbb{R}$ and $a^2+b^2\neq 0$. Suppose $S=\left\{z\in\mathbb{C}: z=\frac{1}{a+ibt},\quad t\in\mathbb{R}, t\neq 0\right\}$, where $i=\sqrt{-1}$. If $z=x+iy$ and $z\in S$, then (x, y) lies on

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a},0\right)$ for $a>0$, $b\neq 0$.

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a},0\right)$ for $a<0$, $b\neq 0$.

(C) the x-axis for $a\neq 0$, $b=0$.

(D) the y-axis for $a=0$, $b\neq 0$.

Solution

(A), (C), (D)

We can write as

$$z=\frac{1}{a-ibt}=\frac{a-ibt}{(a+ibt)(a-ibt)}$$

$$z=\frac{a-ibt}{a^2+b^2t^2}=\left(\frac{a}{a^2+b^2t^2}\right)=i\left(\frac{-bt}{a^2+b^2t^2}\right)=x+iy$$

That is, $x=\frac{a}{a^2+b^2t^2}$ $y=\frac{-bt}{a^2+b^2t^2}$

Therefore, $\frac{y}{x}=\frac{-b}{a}t\Rightarrow t=\frac{-ay}{bx}$

The locus of x:

$$y(a^2+b^2t^2)=-bt$$

$$y\left(a^2+b^2\frac{a^2y^2}{b^2x^2}\right)=\frac{+bay}{bx}$$

$$y\frac{a^2}{x^2}(x^2+y^2)=\frac{ay}{x}$$

$$x^2+y^2=\frac{x}{a}$$

$$x^2+y^2-\frac{x}{a}=0 \text{ (equation of circle)}$$

Therefore, for $a>0$ and $b\neq 0$:

Centre of the circle: $\left(\frac{1}{2a},0\right)$

Radius of the circle: $= \frac{1}{2a}$

Hence, option (A) is correct.

For x-axis:

$$y = 0 = \frac{-bt}{a^2 + b^2t^2} \ (b = 0, a \neq 0)$$

Hence, option (C) is correct.

For y axis:

$$x = 0 = \frac{a}{a^2 + b^2t^2} \ (a = 0, b \neq 0)$$

Hence, option (D) is correct.

13. Let $a, \lambda, m \in \mathbb{R}$. Consider the system of linear equations

$ax + 2y = \lambda$

$3x - 2y = \mu$

Which of the following statement(s) is(are) correct?

(A) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ.

(B) If $a \neq -3$, then the system has a unique solution for all values of λ and μ.

(C) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$.

(D) If $\lambda + \mu \neq 0$, then the system has no solution for $a = 3$.

Solution

(B), (C), (D)

The given system of linear equation is

$$ax + 2y = \lambda$$
$$3x - 2y = \mu$$

By Cramer's rule, we have

$$\Delta = \begin{vmatrix} a & 2 \\ 3 & -2 \end{vmatrix} = -2a - 6 = -2(a+3)$$

$$\Delta_1 = \begin{vmatrix} \lambda & 2 \\ \mu & -2 \end{vmatrix} = -2\lambda - 2\mu = -2(\lambda + \mu)$$

$$\Delta_2 = \begin{vmatrix} a & \lambda \\ 3 & \mu \end{vmatrix} = (a\mu - 3\lambda)$$

For unique solution: $\Delta \neq 0 \Rightarrow a + 3 \neq 0 \Rightarrow a \neq -3$, where λ and μ can take any values.

Hence, option (B) is correct.

For infinite solution: $\Delta = 0$, Δ_1 and Δ_2 are zero.

That is, $a = -3$ and $\lambda + \mu = 0$ or $a\mu - 3\lambda = 0$.

Hence, options (C) are correct.

For no solution: $\Delta = 0$ and either Δ_1 or Δ_2 is non-zero.

That is, $a = -3$ and $\lambda + \mu \neq 0$.

Hence, option (D) is correct.

14. Let $f : \left[-\frac{1}{2}, 2\right] \to \mathbb{R}$ and $g : \left[-\frac{1}{2}, 2\right] \to \mathbb{R}$ be function defined by $f(x) = [x^2 - 3]$ and $g(x) = |x|f(x) + |4x - 7|f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y \in \mathbb{R}$. Then

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$.

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$.

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$.

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$.

Solution

(B), (C)

It is given that

$$f : \left[\frac{1}{2}, 2\right] \to \mathbb{R}$$

$$g : \left[-\frac{1}{2}, 2\right] \to \mathbb{R}$$

Therefore, $f(x) = (x^2 - 3),\ g(x) = (|x| + |14x - 7|)\, f(x)$

$f(x) = (x^2) - 3$

$$f(x) = (x^2) - 3 = \begin{cases} -3 & x \in \left[-\frac{1}{2}, 0\right) \\ -3 & x \in [0, 1) \\ -2 & x \in [1, \sqrt{2}) \\ -1 & x \in [\sqrt{2}, \sqrt{3}) \\ 0 & x \in [\sqrt{3}, 2) \\ 1 & x = 3 \end{cases} = \begin{cases} -3, & x \in \left[-\frac{1}{2}, 1\right) \\ -2, & x \in [1, \sqrt{2}) \\ -1, & x \in [\sqrt{2}, \sqrt{3}) \\ 0, & x \in [\sqrt{3}, 2) \\ 1, & x = 2 \end{cases}$$

$$g(x) = \underbrace{(|x| + |4x - 7|)}_{h(x)} \cdot f(x)$$

Let $h(x) = |x| + |4x - 7|$

$g(x) = h(x) \cdot f(x)$

where $h(x)$ is continuous for all x and it has sharp edge at 0 and 7/4; $f(x)$ is discontinuous at

$x = 1, \sqrt{2}, \sqrt{3}, 2$

Hence, option (B) is correct.

Thus, $g(x)$ is non-differentiable at

$x = 0, 1, \sqrt{2}, \sqrt{3}, 2$

Hence, option (C) is correct.

Paragraph Type

This section contains 2 paragraphs. Based on each paragraph, there are 2 questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

Paragraph for Questions 15 and 16: Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$, respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2, respectively, after two games

15. $P(X > Y)$ is

(A) $\frac{1}{4}$ (B) $\frac{5}{12}$

(C) $\frac{1}{2}$ (D) $\frac{7}{12}$

Solution

(B)

Common data for Questions 15 and 16:

- Probability of wining of $\boldsymbol{T}_1$ against $\boldsymbol{T}_2$ is = 1/2.
- Probability of drawing of $\boldsymbol{T}_1$ against $\boldsymbol{T}_2$ is = 1/6.
- Probability of losing of $\boldsymbol{T}_1$ against $\boldsymbol{T}_2$ is = 1/3.
- 3 points for win.
- 1 point for draw.
- 0 point for loss.

Solution for Question 15:

Now,

$P(x > y) = P(T_1 \text{ wins both game})$
$+ P(T_1 \text{ wins one game and one draw})$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times {}^2C_1 \times \frac{1}{6} = \frac{1}{4} + \frac{1}{6} = \frac{3+2}{12} = \frac{5}{12}$$

16. $P(X = Y)$ is

(A) $\frac{11}{36}$ (B) $\frac{1}{3}$

(C) $\frac{13}{36}$ (D) $\frac{1}{2}$

Solution

(C)

Using the data given, we get

$P(x = y) = P(\text{both team win 1 game or both games draw})$

$$= {}^2C_1 \times \frac{1}{2} \times \frac{1}{3} + \frac{1}{6} \times \frac{1}{6} = \frac{13}{36}$$

Paragraph for Questions 17 and 18: Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

17. The orthocentre of the triangle F_1MN is

(A) $\left(-\frac{9}{10}, 0\right)$ (B) $\left(\frac{2}{3}, 0\right)$

(C) $\left(\frac{9}{10}, 0\right)$ (D) $\left(\frac{2}{3}, \sqrt{6}\right)$

Solution

(A)

Common data for Questions 17 and 18:

It is given that $F_1(x, 0)$ and $F_2\ (x_2, 0)$ are the foci of the ellipse:

$$\frac{x^2}{9}+\frac{y^2}{8}=1$$

Therefore, $a^2 = 9$ and $b^2 = 8$.

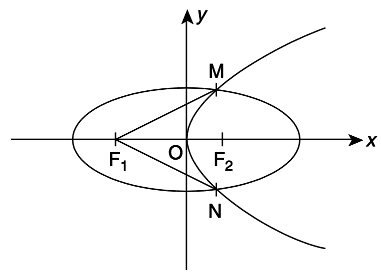
$$b^2 = a^2\,(1 - e^2)$$

$$\frac{8}{9}1-e^2 \Rightarrow e^2 = 1-\frac{8}{9}=\frac{1}{9} \Rightarrow e = \frac{1}{3}$$

The focus is

$$F_1\left(-3\times\frac{1}{3}, 0\right) \text{ and } F_2\left(3\times\frac{1}{3}, 0\right)$$

That is, $F_1(-1, 0)$ and $F_2(1, 0)$.



The equation of parabola is

$$y^2 = 4(\text{OF}_2)x$$

$$y^2 = 4x\ (\text{OF}_2 = 1)$$

The point of intersection of ellipse and parabola is

$$\frac{x^2}{9}+\frac{4x}{8}=1 \Rightarrow \frac{x^2}{9}+\frac{x}{2}=1$$

$$\Rightarrow 2x^2 + 9x - 18 = 0$$

$$\Rightarrow 2x^2 + 12x - 3x - 18 = 0$$

$$\Rightarrow 2x(x + 6) - 3(x + 6) = 0$$

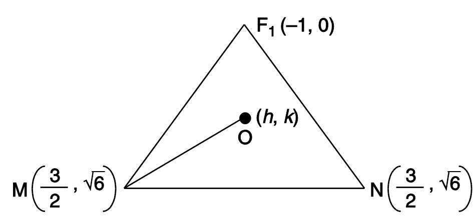
$$\Rightarrow x = \frac{3}{2}\ (x = -6 \text{ is rejected})$$

Now, $y^2(4)\frac{3}{2}=6$

$$y=\pm\sqrt{6}$$

That is, the points M and N are, respectively, $\text{M}\left(\frac{3}{2},\sqrt{6}\right)$ and $\text{N}\left(\frac{3}{2},-\sqrt{6}\right)$.

Solution for Question 17:



Let the orthocentre be (h, k).

The slope of OM $=\dfrac{k-\sqrt{6}}{h-(3/2)}$.

The slope of ON $= \dfrac{\sqrt{6}}{-1-(3/2)} = \dfrac{-2\sqrt{6}}{5}$.

Now, $\left(\dfrac{k-\sqrt{6}}{h-(3/2)}\right)\left(\dfrac{-2\sqrt{6}}{5}\right) = -1$

$$2\sqrt{6}k - 12 = 5h - \frac{15}{2}$$

$$5h - 2\sqrt{6}k = \frac{15}{2} - 12 = \frac{-9}{2}$$

The slope of ON $= \dfrac{k+\sqrt{6}}{h-(3/2)}$.

The slope of F_1M $= \dfrac{\sqrt{6}}{(3/2)+1} = \dfrac{2\sqrt{6}}{5}$.

$$\frac{k+\sqrt{6}}{h-(3/2)} \times \frac{2\sqrt{6}}{5} = -1$$

$$2\sqrt{6}k + 12 = -5h + \frac{15}{2}$$

$$5h + 2\sqrt{6}k = \frac{15}{2} - 12 = \frac{-9}{2}$$

$$5h + 2\sqrt{6}k = \frac{-9}{2} \quad (1)$$

$$5h - 2\sqrt{6}k = \frac{-9}{2} \quad (2)$$

Solving Eqs. (1) and (2), we get

$$10h = -9 \Rightarrow h = \frac{-9}{10} \text{ and } k = 0$$

Hence, the orthocentre of the triangle F_1MN is $\left(-\dfrac{9}{10}, 0\right)$.

18. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the *x*-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is

(A) 3:4 (B) 4:5

(C) 5:8 (D) 2:3

Solution

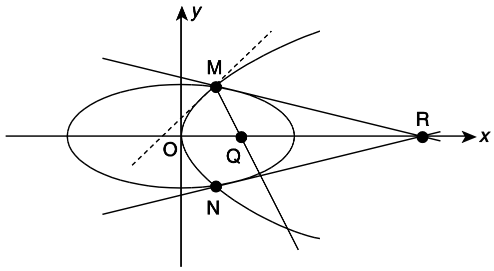
(C)

From the common data given in the Solution of Question 17, the equation of tangent to ellipse at *m* is

$m\left(\dfrac{3}{2}, \sqrt{6}\right)$

Now, $\dfrac{3x}{2(9)} + \dfrac{y\sqrt{6}}{8} = 1$ meets the *x*-axis, $y = 0$.

That is, $\dfrac{x}{6} = 1 \Rightarrow x = 6$.



The Point R is (6, 0).
The normal to the parabola at *m* is

$$y + tx = 2at + at^3$$

where $a = 1$, $t^2 = \frac{3}{2}$ and $2t = \sqrt{6}$. Therefore, $t = \sqrt{\frac{3}{2}}$.

Now, $y + \sqrt{\frac{3}{2}}x = 2\sqrt{\frac{3}{2}} + \frac{3}{2}\sqrt{\frac{3}{2}}$

which cuts the *x*-axis at $y = 0$.

$$\sqrt{\frac{3}{2}}x = 2\sqrt{\frac{3}{2}} + \frac{3}{2}\sqrt{\frac{3}{2}}$$

$$x = 2 + \frac{3}{2} = \frac{7}{2}$$

That is, we have $Q\left(\frac{7}{2}, 0\right)$.

The area of MQR is

$$\left|\frac{1}{2}\right| \begin{vmatrix} 3/2 & \sqrt{6} & 1 \\ 6 & 0 & 1 \\ 7/2 & 0 & 1 \end{vmatrix} = \left|\frac{\sqrt{6}}{2}\left(6 - \frac{7}{2}\right)\right| = \frac{5\sqrt{6}}{4}$$

The area of the quadrilateral MF_1NF_2 is

$2(\Delta m_1F_1F_2) = 2\sqrt{6}$

and the required ratio is

$$\frac{5\sqrt{6}}{4.2\sqrt{6}} = \frac{5}{8}$$