JEE ADVANCED 2017
PAPER 1
MATHEMATICS

One or More Than One Option Correct Type

This section contains 7 questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONE OR MORE** is(are) correct.

1. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following CANNOT be sides of a right-angled triangle?

(A) a, 4, 1 (B) a, 4, 2
(C) $2a$, 8, 1 (D) $2a$, 4, 1

Solution

(A), (B), (C) Line $y = mx + c$ is the tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if following condition is satisfied:

$$c^2 = a^2m^2 - b^2 \quad (1)$$

It is given that the equation of line is

$$2x - y + 1 = 0$$

and hyperbola is $$\frac{x^2}{a^2} - \frac{y^2}{16} = 1$$

Let us consider the equation

$$2x - y + 1 = 0$$

Therefore, $$y = 2x + 1 \Rightarrow m = 2 \text{ and } c = 1$$

From the equation of hyperbola, we get $b^2 = 16$ and a^2 as follows:

Substituting all values in Eq. (1), we get

$$1^2 = a^2(2)^2 - 16$$

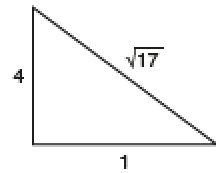
$$\Rightarrow 1 = 4a^2 - 16$$

$$\Rightarrow 4a^2 = 16 + 1 \Rightarrow 4a^2 = 17$$

$$\Rightarrow a^2 = \frac{17}{4} \Rightarrow a = \pm\sqrt{\frac{17}{4}} = \pm\frac{\sqrt{17}}{2}$$

$$\Rightarrow a = \pm\frac{\sqrt{17}}{2}$$

- **For option (A):** a, 4, 1 $\Rightarrow \pm\frac{\sqrt{17}}{2}, 4, 1$ is not a right-angled triangle $\left(\text{since } \sqrt{4^2 + 1^2} \neq \frac{\sqrt{17}}{2}\right)$
- **For option (B):** a, 4, 2 $\Rightarrow \pm\frac{\sqrt{17}}{2}, 4, 2$ is not a right-angled triangle $\left(\text{since } \sqrt{4^2 + 2^2} \neq \frac{\sqrt{17}}{2}\right)$
- **For option (C):** $2a$, 8, 1 $\Rightarrow \pm\sqrt{17}, 8, 1$ is not a right-angled triangle $(\text{since } \sqrt{8^2 + 1^2} \neq \sqrt{17})$
- **For option (D):** $2a$, 4, 1 $\Rightarrow \pm\sqrt{17}, 4, 1$ is a right-angled triangle $\because \sqrt{4^2 + 1^2} = \sqrt{16 + 1} = \sqrt{17}$



Hence, the correct options are (A), (B) and (C).

2 If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k?

(A) $p = -2, h = 2, k = -4$ (B) $p = -1, h = 1, k = -3$
(C) $p = 2, h = 3, k = -4$ (D) $p = 5, h = 4, k = -3$

Solution

(C) It is given that the equation of parabola, $y^2 = 16x$ and the equation of the chord is

$$2x + y = p \quad (1)$$

Equation of chord with middle point (h, k) is given as

$$ky - 16\left(\frac{x+h}{2}\right) = k^2 - 16h$$

$$\Rightarrow ky - 8(x + h) = k^2 - 16h$$

$$\Rightarrow ky - 8x - 8h = k^2 - 16h$$

$$\Rightarrow ky - 8x - 8h - k^2 + 16h = 0$$

$$\Rightarrow -8x + ky + 8h - k^2 = 0$$

$$\Rightarrow 8x - ky = 8h - k^2 \quad (2)$$

Dividing Eq. (2) by 4, we get

$$\frac{8x}{4} - \frac{ky}{4} = \frac{8h}{4} - \frac{k^2}{4}$$

$$\Rightarrow 2x - \frac{ky}{4} = 2h - \frac{k^2}{4}$$

On comparing this with Eq. (1), we get

$$\frac{-k}{4} = 1 \text{ and } p = 2h - \frac{k^2}{4}$$

$$\Rightarrow k = -4 \text{ and } p = 2h - \frac{(-4)^2}{4} = 2h - 4$$

$$\Rightarrow p = 2h - 4$$

$$\Rightarrow 2h - p = 4$$

Only the values $p = 2$ and $h = 3$ satisfy this equation. Therefore,

$$p = 2, h = 3 \text{ and } k = -4$$

Hence, only option (C) is correct.

3. Let $[x]$ be the greatest integer less than or equals to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous?

(A) $x = -1$ (B) $x = 0$
(C) $x = 1$ (D) $x = 2$

Solution

(A), (C), (D) It is given that

$$f(x) = x\cos(\pi(x + [x]))$$

and $$f(x) = (-1)^{[x]}\, x\cos\pi x$$

This function is discontinuous at all integral points except $x = 0$. At $x = z$; $z = 0, \pm1, \pm2, \ldots$

$$f(z) = z\cos\pi(2z) = z$$
$$f(z^+) = z\cos\pi(2z) = z$$
$$f(z^-) = z\cos\pi(2z - 1) = -z$$
$$f(0) = 0$$

Hence, the correct options are (A), (C) and (D).

4. Let $f\colon \mathbb{R} \to (0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1)?

(A) $x^9 - f(x)$ (B) $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\, dt$

(C) $e^x - \int_0^x f(t)\sin t\, dt$ (D) $f(x) + \int_0^{\pi/2} f(t)\sin t\, dt$

Solution

(A), (B) We discuss the options as follows:

- **Option (A):** Let $g(x) = x^9 - f(x)$.

$$g(0) = -f(0) < 0 \qquad [\text{as } f \in (0,1)]$$
$$g(1) = 1^9 - f(1) = 1 - f(1) > 0 \qquad [\text{as } f \in (0,1)]$$

Hence, option (A) is correct.

- **Option (B):** Let $g(x) = x - \int_0^{\pi/2-x} f(t)\cos t\, dt$.

$$g(0) = 0 - \int_0^{\pi/2-0} f(t)\cos t\, dt < 0$$

and $$g(1) = 1 - \int_0^{\pi/2-1} f(t)\cos t\, dt > 0$$

Hence, option (B) is correct.

- **Option (C):** Let $g(x) = e^x - \int_0^x f(t)\sin t\, dt$.

Differentiating w.r.t. x, we get

$$g'(x) = e^x - f(x)\sin x \qquad (1)$$

Now $g(0) = 1$

Also, we know that $f(x) \in (0,1) \Rightarrow 0 < f(x) < 1 \Rightarrow 0 < f(x)\sin x < 1$

Therefore, from Eq. (1), we get

$$g'(x) > 0$$

Thus, $g(x)$ is strictly an increasing function.

Hence, option (C) is incorrect.

- **Option (D):** Let $g(x) = f(x) + \int_0^{\pi/2} f(t)\sin t\, dt$

  Since $f(x) \in (0, 1)$, we get $g(x) > 0$.

  Hence, option (D) is incorrect.

5. Which of the following is(are) NOT the square of a 3 × 3 matrix with real entries?

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Solution

(B), (D) For a matrix to be a square of matrix with real entries, its determinant should be positive.

- **Option (A):** The determinant $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is possible: 1(1) – 0(0) + 0(0) = +1.
- **Option (B):** The determinant $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is not possible: 1(−1) – 0(0) + 0(0) = −1.
- **Option (C):** The determinant $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is possible: 1(1) – 0(0) + 0(0) = +1.
- **Option (D):** The determinant $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is not possible: −1(1) – 0(0) + 0(0) = −1.

Hence, options (B) and (D) are correct due to the reason that they are NOT the square of a 3 × 3 matrix with real entries.

6. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x?

(A) $-1+\sqrt{1-y^2}$ (B) $-1-\sqrt{1-y^2}$

(C) $1+\sqrt{1+y^2}$ (D) $1-\sqrt{1+y^2}$

Solution

(A), (B) It is given that $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$.

Therefore,

$$\text{Im}\left(\frac{a(x+iy)+b}{(x+iy)+1}\right) = y$$

$$\Rightarrow \text{Im}\left(\frac{ax+iay+b}{x+iy+1}\right)=y$$

Rationalising the above equation: Multiplying and dividing LHS by (x + 1 – iy), we get

$$\text{Im}\left(\frac{(ax+iay+b)}{(x+iy+1)}\times\frac{(x+1-iy)}{(x+1-iy)}\right)=y$$

Using $a^2 - b^2 = (a + b)(a - b)$, we get

$$\text{Im}\left(\frac{(ax+iay+b)(x+1-iy)}{(x+1)^2-(iy)^2}\right)=y$$

$$\text{Im}\left(\frac{ax^2+ax-iayx+iaxy-i^2ay^2+bx+b-iby}{(x+1)^2+y^2}\right)=y$$

(as $i^2=-1$)

$$\text{Im}\left(\frac{(ax^2+ax+ay^2+bx+b)+i(axy-ayx+ay-by)}{(x+1)^2+y^2}\right)=y$$

Rearranging LHS, we get

$$\text{Im}\left(\frac{[(ax^2+bx)+(ax+b)+ay^2]+i(ay-by)}{(x+1)^2+y^2}\right)=y$$

$$\Rightarrow \frac{ay-by}{(x+1)^2+y^2}=y$$

(as Im of the value in bracket is coefficient of i)

$$\Rightarrow \not{y}(a-b)=\not{y}((x+1)^2+y^2)\Rightarrow(a-b)=(x+1)^2+y^2$$

It is given that $a - b = 1$ and $y \neq 0$. Therefore,

$$1 = (x + 1)^2 + y^2$$

$$\Rightarrow 1 - y^2 = (x + 1)^2$$

$$\Rightarrow (x+1)=\pm\sqrt{1-y^2} \qquad (\text{as } x^2=b\Rightarrow x=\pm\sqrt{b})$$

$$\Rightarrow 1=-1\pm\sqrt{1-y^2}$$

or $\quad x=-1+\sqrt{1-y^2}$ and $x=-1-\sqrt{1-y^2}$

Hence, options (A) and (B) are correct.

7. Let X and Y be two events such that $P(X)=\frac{1}{3}$, $P(X|Y)=\frac{1}{2}$ and $P(Y|X)=\frac{2}{5}$. Then

(A) $P(Y)=\frac{4}{15}$ (B) $P(X'|Y)=\frac{1}{2}$

(C) $P(X\cap Y)=\frac{1}{5}$ (D) $P(X\cup Y)=\frac{2}{5}$

Solution

(A), (B) It is given that

$$P(X)=\frac{1}{3} \qquad (1)$$

$$P(X \mid Y)=\frac{1}{2}$$

and $$P(Y \mid X)=\frac{2}{5}$$

Now, $$P(X \mid Y)=\frac{P(X \cap Y)}{P(Y)}=\frac{1}{2} \qquad (2)$$

where $\cap$ denotes intersection and

$$P(Y \mid X)=\frac{P(Y \cap X)}{P(X)}=\frac{2}{5} \qquad (3)$$

From Eq. (2), we get

$$P(X \cap Y)=\frac{P(Y)}{2}$$

From Eq. (3), we get

$$P(Y \cap X)=\frac{2}{5}P(X)=\frac{2}{5}\times\frac{1}{3}=\frac{2}{15}$$

also $$P(X \cap Y)=P(Y \cap X)=\frac{2}{15}$$

$$\Rightarrow \frac{P(Y)}{2}=\frac{2}{15} \Rightarrow \; P(Y)=\frac{4}{15}$$

Hence, option (A) is correct.

Now $$P(X' \mid Y)=\frac{P(X' \cap Y)}{P(Y)}=\frac{P(Y)-P(X \cap Y)}{P(Y)}$$

Substituting all values, we get

$$P(X' \mid Y)=\frac{\left(\frac{4}{15}-\frac{2}{15}\right)}{4/15}=\frac{\left(\frac{4-2}{15}\right)}{4/15}=\frac{2}{4}=\frac{1}{2}$$

$$\Rightarrow \; P(X' \mid Y)=\frac{1}{2}$$

Hence, option (B) is correct.
We know that

$$P(X \cap Y)=\frac{2}{15}$$

Hence, option (C) is incorrect.
Now $$P(X \cup Y)=P(X)+P(Y)-P(X \cap Y)$$

Substituting all values, we get

$$P(X \cup Y)=\frac{1}{3}+\frac{4}{15}-\frac{2}{15}=\frac{1}{3}+\frac{2}{15}=\frac{5+2}{15}=\frac{7}{15}$$

$$\Rightarrow P(X \cup Y)=\frac{7}{15}$$

Hence, option (D) is incorrect.

<H2>Integer Answer Type

This section contains 5 questions. The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive.

8. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points?

Solution

We consider the following three cases:

- **Case 1:** The circle passes through the origin, that is, $p = 0$; now, the equation of circle becomes

$$x^2 + y^2 + 2x + 4y = 0$$

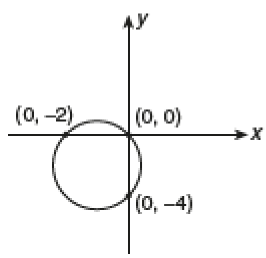
Now, $x = 0$. Therefore,

$$y^2 + 4y = 0$$
$$\Rightarrow y(y + 4) = 0$$

Therefore, $y = 0$ and $y = -4$.

Thus, $y = 0$ gives

$$x^2 + 2x = 0$$
$$\Rightarrow x(x + 2) = 0$$
$$\Rightarrow x = 0, -2$$



- **Case 2:** When the circle touches y-axis, then the circle intersects x-axis at two distinct points.

Substituting $y = 0$ in the equation of circle, we get

$$x^2 + 2x - p = 0$$

Now, from $g^2 - c > 0$ and, we get

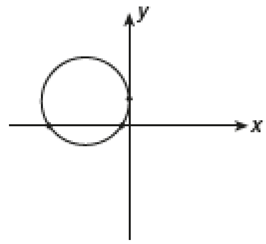
$$1^2 - (-p) > 0 \text{ and } 2^2 - (-p) = 0$$
$$1 + p > 0 \text{ and } 4 + p = 0$$
$$p > -1 \text{ and } p = -4$$

which is a contradiction. Also, for $p = -4$, we get

$$x^2 - 2x + 4 = 0$$

Therefore, $$x = \frac{2 \pm \sqrt{4 - 4 \times 4}}{2} = \frac{2 \pm 2\sqrt{3}i}{2} = 1 \pm \sqrt{3}i$$

which are imaginary roots. Therefore, this Case 2 is not possible.



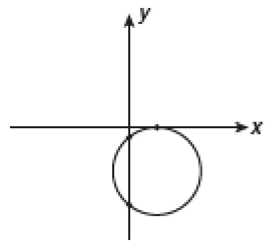
- **Case 3:** When the circle touches x-axis, the circle intersects y-axis at two distinct points. Substituting $x = 0$, we get

$$y^2 + 4y - p = 0$$

Now, from $g^2 - c = 0$ and $f^2 - c > 0$, we have

$$1^2 - (-p) = 0 \text{ and } 2^2 = (-p) > 0$$

$$p = -1 \text{ and } 4 + p > 0 \Rightarrow p > -4$$



Therefore, the equation of circle becomes

$$y^2 + 4y + 1 = 0$$

$$\Rightarrow \quad y = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2}$$

$$y = \frac{-4 \pm \sqrt{4 \times 3}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

which are real values. Therefore, Case 3 is possible.

Thus, for two values of p ($p = 0$ and $p = -1$), the circle and the coordinate axes have exactly three points in common.

Hence, the correct answer is 2.

9. Let f: $\mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$. If $g(x) = \int_x^{\pi/2} [f'(t) \operatorname{cosec} t - \cot t \operatorname{cosec} t \; f(t)] dt$ for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x \to 0} g(x)$ = ______.

Solution

It is given that

$$g(x) = \int_x^{\pi/2} [f'(t) \operatorname{cosec} t - \cot t \operatorname{cosec} t \; f(t)] dt$$

We know that

$$\frac{d}{dt}(f(t) \operatorname{cosec} t) = f'(t) \operatorname{cosec} t - f(t) \operatorname{cosec} t \cos t$$

Therefore, $$g(x)=\int_x^{\pi/2}\frac{d}{dt}(f(t)\text{cosec } t)dt$$

$$\Rightarrow\ g(x)=f(t)\text{ cosec } t\Big|_x^{\pi/2}$$

$$g(x)=f\left(\frac{\pi}{2}\right)\text{cosec}\left(\frac{\pi}{2}\right)-f(x)\text{ cosec}x$$

It is given that $f(\pi/2)=3$ and $\text{cosec}(\pi/2)=1$. Therefore,

$$g(x)=3-f(x)\text{ cosec } x=\frac{3-f(x)}{\sin x}$$

Now, $$\lim_{x\to 0}g(x)=\lim_{x\to 0}\left(\frac{3-f(x)}{\sin x}\right)=3-\lim_{x\to 0}\frac{f(x)}{\sin x}$$

Since $f(0)=0$ and $\sin 0=0$, we get

$$\lim_{x\to 0}\frac{f(x)}{\sin x}=\frac{0}{0}$$

Taking derivative: Using $\frac{d}{dx}\sin x=\cos x$, we get

$$\lim_{x\to 0}\frac{f'(x)}{\cos x}=\frac{1}{1} \qquad [\text{as } f'(0)=1\text{ given}]$$

Therefore, $$\lim_{x\to 0}g(x)=3-1=2.$$

10. For a real number α, if the system

$$\begin{bmatrix}1 & \alpha & \alpha^2\\ \alpha & 1 & \alpha\\ \alpha^2 & \alpha & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1+\alpha+\alpha^2=$ _______.

Solution

It is given that

$$\begin{bmatrix}1 & \alpha & \alpha^2\\ \alpha & 1 & \alpha\\ \alpha^2 & \alpha & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}$$

Therefore,

$$\begin{vmatrix}1 & \alpha & \alpha^2\\ \alpha & 1 & \alpha\\ \alpha^2 & \alpha & 1\end{vmatrix}=0$$

$$\Rightarrow 1(1-\alpha^2)-\alpha(\alpha-\alpha^3)+\alpha^2(\alpha^2-\alpha^2)=0$$

$$\Rightarrow\ (1-\alpha^2)-\alpha^2(1-\alpha^2)=0\ \Rightarrow\ (1-\alpha^2)(1-\alpha^2)=0$$

$$\Rightarrow\ (1-\alpha^2)^2=0$$

$$\Rightarrow\ \alpha^2=1\Rightarrow\alpha=\pm 1$$

For $\alpha = 1$, the given system of linear equations has no solution. That is,

$$\begin{bmatrix} +1 & +1 & +1 \\ +1 & +1 & +1 \\ +1 & +1 & +1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$x + y + z = 1$$
$$x + y + z = -1$$
$$x + y + z = 1$$

Since two planes are parallel, $\alpha = 1$ is rejected and for $\alpha = -1$, the given system of linear equations has coincident planes.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} x - y + z = 1 \\ -x + y - z = -1 \Rightarrow x - y + z = 1 \\ x - y + 1 = 1 \end{cases}$$

Therefore, $\alpha = -1$ holds well. Therefore,

$$1 + \alpha + \alpha^2 = 1 + (-1) + (-1)^2$$
$$= 1 - 1 + 1 = 1$$

11. Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I, J. Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then, $\frac{y}{9x} =$ ______.

Solution The given, formed word is of length 10.

It is given that x is the number of words where no letter is repeated.

Also, it is given that y is the number of words where exactly one letter is repeated twice and no other letter is repeated. Therefore,

$$x = 10!$$

and $$y = {}^{10}C_1 \times {}^{10}C_2 \times {}^{9}C_8 \times 8!$$

Thus, $$\frac{y}{9x} = \frac{{}^{10}C_1 \times {}^{10}C_2 \times {}^{9}C_8 \times 8!}{9 \times 10!}$$

Using ${}^{n}C_r = \frac{n!}{r!(n-r)!}$, we get

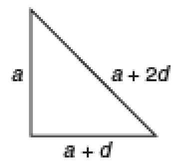
$$\frac{y}{9x} = \frac{\frac{10!}{1!(10-1)!} \times \frac{10!}{2!(10-2)!} \times \frac{9!}{8!(9-8)!} \times 8!}{9 \times 10!}$$

$$= \frac{10!}{9 \times 2! \times 8!} = \frac{10 \times 9 \times 8!}{9 \times 2 \times 8!} = \frac{10}{2} = 5$$ [Using $n! = n(n-1)(n-2)$

….1!]

12. The sides of a right-angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side?

Solution Let a be the first term and d be the common difference, then the sides of triangle are a, $a + d$ and $a + 2d$.



$$\text{Area of triangle} = \frac{1}{2} \times \text{Base} \times \text{Height}$$

Substituting the values, we get

$$24 = \frac{1}{2} \times (a+d) \times a$$

$$a(a+d) = 2 \times 24$$

$$a(a+d) = 48 \qquad (1)$$

Applying Pythagoras theorem, we get

$$a^2 + (a + d)^2 = (a + 2d)^2$$

$$\Rightarrow a^2 + a^2 + d^2 + 2ad = a^2 + 4d^2 + 4ad$$

$$\Rightarrow a^2 + 4d^2 + 4ad - a^2 - a^2 - d^2 - 2ad = 0$$

$$\Rightarrow 3d^2 - a^2 + 2ad = 0 \Rightarrow 3d^2 + 3ad - ad - a^2 = 0$$

$$\Rightarrow 3d(d + a) - a(d + a) = 0$$

$$\Rightarrow (3d - a)(a + d) = 0$$

From Eq. (1), we know that $a + d \neq 0$. Therefore,

$$3d - a = 0 \Rightarrow 3d = a$$

Substituting in Eq. (1), we get

$$3d(3d + d) = 48$$

$$\Rightarrow 3d \times 4d = 48$$

$$\Rightarrow d^2 = \frac{48}{12} \Rightarrow d^2 = 4 \Rightarrow d = 2$$

From $3d = a$, we get

$$a = 2 \times 3 = 6 \Rightarrow a = 6$$

Therefore, the sides of the triangle are 6, (6 + 2), (6 + 2 × 2), that is, 6, 8, 10.

Thus, the length of the smallest side is 6.

<H2>Matrix-Match Type

This section contains 6 questions of matching type. There are two tables (each having 3 columns and 4 rows). Based on each table, there are 3 questions. Each question has 4 options (A), (B), (C) and (D), **ONLY ONE** of these four options is correct.

Directions for Qns. 13, 14 and 15: Answer the questions by appropriately matching the information given in the three columns of the following table:

Columns 1, 2 and 3 contain conics, equations of tangents to the conics and points of contact, respectively.		
Column 1	**Column 2**	**Column 3**

(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $x\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

13. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (−1, 1), then which of the following options is the only CORRECT combination for obtaining its equation?

(A) (I) (i) (P)
(B) (I) (ii) (Q)
(C) (II) (ii) (Q)
(D) (III) (i) (P)

Solution

(B) We discuss the combinations as follows:

- For the given equation $a = \sqrt{2}$ and the point of contact (−1, 1), the conic equation $x^2 + y^2 = a^2$ satisfies these conditions [i.e. (I) of Column 1].
- Thus, the equation of tangent is $-x + y = 2$ that is, $y = x + 2$; thus, (ii) of Column 2 satisfies this condition with $m = 1$. That is, $y = mx + a\sqrt{m^2+1}$.
- It is given that the point of constant (−1, 1); given $m = 1$ and $a = \sqrt{2}$, we see that (Q) of Column 3 $\left[\text{i.e.}\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)\right]$ gives the point of contact (−1, 1).

Hence, the correct combination is (I) (ii) (Q); thus, option (B) is correct.

14. If a tangent to a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is (8, 16), then which of the following options is the only CORRECT combination?

(A) (I) (ii) (Q)
(B) (II) (iv) (R)
(C) (III) (i) (P)
(D) (III) (ii) (Q)

Solution

(C) We discuss the combinations as follows:

- It is given that the tangent to the conic is $y = x + 8$ and the point of contact is (8, 16).
- On comparing equation $y = x + 8$ with the options provided in Column 2 containing equations of tangents, we get $m = 1$.
- Comparing $y = x + 8$ with option (i) of Column (2), we get $a = 8$.
- The given point of contact (8, 16) and $a = 8$, we see that (III) of Column 1 (i.e. $y^2 = 4ax$) is satisfied with these conditions.
- It is given that $a = 8$ and $m = 1$, $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ gives (8, 16) as point of contact.

Hence, the correct combination is (III) (i) (P); thus, option (C) is correct.

15. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4,$ then which of the following options is the only CORRECT combination?

(A) (IV) (iii) (S) (B) (IV) (iv) (S)
(C) (II) (iii) (R) (D) (II) (iv) (R)

Solution

(D) It is given that the tangent to conic is

$$\sqrt{3}x + 2y = 4 \qquad (1)$$

and the point of contact is $\left(\sqrt{3}, \frac{1}{2}\right)$. Therefore, from Eq. (1), we get

$$2y = 4 - \sqrt{3}x$$

$$\Rightarrow \; y = \frac{-\sqrt{3}}{2}x + 2 \qquad (2)$$

On comparing this equation with the options provided in Column 2, we get $m = \frac{-\sqrt{3}}{2}$.

Equation (1) can also be written as

$$\sqrt{3}x + 4\left(\frac{1}{2}\right)y = 4$$

On comparing with option (II), we get

$$a^2 = 4 \Rightarrow a = 2$$

Now, for $a = 2$, $m = \frac{-\sqrt{3}}{2}$, $\left(\frac{-a^2 m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$ gives $\left(\sqrt{3}, \frac{1}{2}\right)$ as the point of contact.

Hence, the correct combination is (II) (iv) (R); thus, option (C) is correct.

Directions for Qns. 16, 17 and 18: Answer the questions by appropriately matching the information given in the three columns of the following table:

Let $f(x) = x + \log_e x - x\log_e x,\; x \in (0, \infty)$.

- Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.
- Column 2 contains information about the limiting behaviour of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.
- Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

Column 1	Column 2	Column 3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in (0, 1)
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in (0, 1)
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

16. Which of the following options is the only CORRECT combination?

(A) (I) (i) (P) (B) (II) (ii) (Q)
(C) (III) (iii) (R) (D) (IV) (iv) (S)

Solution

(B) It is given that

$$f(x) = x + \log_e x - x\log_e x, \qquad x \in (0, \infty)$$

$$\Rightarrow f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}(x + \log x - x\log x) = 1 + \frac{1}{x} - \log x - x \cdot \frac{1}{x}$$

$$f'(x) = \frac{1}{x} - \log x$$

$$\Rightarrow f''(x) = \frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{x} - \log x\right) = \frac{-1}{x^2} - \frac{1}{x}$$

- $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x + \log x - x\log x) = -\infty.$ Hence, option (ii) is correct.
- $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \left(\frac{1}{x} - \log x\right) = -\infty.$ Hence, option (iii) is correct.
- $\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} \left(\frac{-1}{x^2} - \frac{1}{x}\right) = 0.$ Hence, option (iv) is correct.

Also, we have

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \left(\frac{1}{x} - \log x\right) = \infty$$

Thus $\lim_{x\to 0^+} f'(x) = \infty$ and $\lim_{x\to\infty} f'(x) = -\infty$ implies $f'(x)$ is decreasing function.

Hence, options (Q) and (S) are correct.

Therefore, from the given options, (II) (ii) (Q) is correct; thus, option (B) is correct.

17. Which of the following options is the only CORRECT combination?

(A) (I) (i) (R) (B) (II) (iii) (S)

(C) (III) (iv) (P) (D) (IV) (i) (S)

Solution

(B) Following equations in the solution of Qn. 16, we have the following conclusions:

- In Column 2, option (i) is false and (ii) (iii) and (iv) are correct
- in Column 3, options (Q) and (S) are correct and options (P) and (R) are false.

Hence, from the given options, only (II) (iii) (S) is correct; thus, option (B) is correct.

18. Which of the following options is the only INCORRECT combination?

(A) (I) (iii) (P) (B) (II) (iv) (Q)

(C) (III) (i) (R) (D) (II) (iii) (P)

Solution

(C) Following explanation of Qn. 16, we have the following conclusions:

- In Column 2, option (i) is false.
- In Column 3, options (P) and (R) are false.

Hence, from given options, only (III) (i) (R) combination is INCORRECT; thus, option (C) is correct.

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