JEE ADVANCED 2017

PAPER 2

MATHEMATICS

One or More Than One Option Correct Type

This section contains 7 questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONE OR MORE** is(are) correct.

1. The equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is

(A) $14x + 2y - 15z = 1$ (B) $14x - 2y + 15z = 27$

(C) $14x + 2y + 15z = 31$ (D) $-14x + 2y + 15z = 3$

Solution

(C) Let the equation of the plane passing through point (1, 1, 1) be

$$a(x-1) + b(y-1) + c(z-1) = 0$$

The normal to the plane is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix} = \hat{i}(-2-12) - \hat{j}(-4+6) + \hat{k}(-12-3) = -14\hat{i} - 2\hat{j} - 15\hat{k}$$

Thus, the equation of plane is

$$-14(x-1) - 2(y-1) - 15(z-1) = 0$$

$$-14x + 14 - 2y + 2 - 15z + 15 = 0$$

$$-14x - 2y - 15z + 31 = 0 \Rightarrow 14x + 2y + 15z = 31$$

Hence, option (C) is correct.

2. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

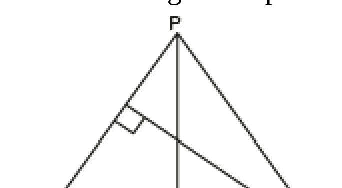
$\overrightarrow{OP}\cdot\overrightarrow{PQ}+\overrightarrow{OR}\cdot\overrightarrow{OS}=\overrightarrow{OR}\cdot\overrightarrow{OP}+\overrightarrow{OQ}\cdot\overrightarrow{OS}=\overrightarrow{OQ}\cdot\overrightarrow{OR}+\overrightarrow{OP}\cdot\overrightarrow{OS}$. Then the triangle PQR has S as its

(A) centroid. (B) circumcenter.

(C) incentre. (D) orthocenter.

Solution

(D) The given geometrical situation for the triangle is depicted in the following figure:



It is given that

$$\overrightarrow{OP}\cdot\overrightarrow{OQ}+\overrightarrow{OR}\cdot\overrightarrow{OS}=\overrightarrow{OR}\cdot\overrightarrow{OP}+\overrightarrow{OQ}\cdot\overrightarrow{OS}=\overrightarrow{OQ}\cdot\overrightarrow{OR}+\overrightarrow{OP}\cdot\overrightarrow{OS}$$

Let us consider that

$$\overrightarrow{OP}\cdot\overrightarrow{OQ}+\overrightarrow{OR}\cdot\overrightarrow{OS}=\overrightarrow{OR}\cdot\overrightarrow{OP}+\overrightarrow{OQ}\cdot\overrightarrow{OS}$$

That is,

$$\overrightarrow{OP}\cdot\overrightarrow{OQ}+\overrightarrow{OR}\cdot\overrightarrow{OS}-\overrightarrow{OR}\cdot\overrightarrow{OP}-\overrightarrow{OQ}\cdot\overrightarrow{OS}=0$$

$$\Rightarrow \overrightarrow{OP}\cdot(\overrightarrow{OQ}-\overrightarrow{OR})+\overrightarrow{OS}\cdot(\overrightarrow{OR}\cdot\overrightarrow{OQ})=0$$

$$\Rightarrow (\overrightarrow{OP} - \overrightarrow{OS}) \cdot (\overrightarrow{OQ} - \overrightarrow{OR}) = 0$$

Therefore, $\overrightarrow{SP} \cdot \overrightarrow{RQ} = 0 \;\Rightarrow\; \overrightarrow{SP} \perp \overrightarrow{RQ}$

Similarly, let us consider

$$\overrightarrow{OR} \cdot \overrightarrow{OP} + \overrightarrow{OQ} \cdot \overrightarrow{OS} = \overrightarrow{OQ} \cdot \overrightarrow{OR} + \overrightarrow{OP} \cdot \overrightarrow{OS}$$

$$\Rightarrow \overrightarrow{OR} \cdot \overrightarrow{OP} + \overrightarrow{OQ} \cdot \overrightarrow{OS} - \overrightarrow{OQ} \cdot \overrightarrow{OR} - \overrightarrow{OP} \cdot \overrightarrow{OS} = 0$$

$$\Rightarrow \overrightarrow{OR} \cdot (\overrightarrow{OP} - \overrightarrow{OQ}) + \overrightarrow{OS} \cdot (\overrightarrow{OQ} - \overrightarrow{OP}) = 0$$

$$\Rightarrow (\overrightarrow{OR} - \overrightarrow{OS}) \cdot (\overrightarrow{OP} - \overrightarrow{OQ}) = 0$$

$$\Rightarrow \overrightarrow{SR} \cdot \overrightarrow{QP} = 0$$

$$\Rightarrow \overrightarrow{SR} \perp \overrightarrow{QP}$$

Hence, point S is orthocentre of the triangle PQR.

3. If $y = y(x)$ satisfies the differential equation $8\sqrt{x}(\sqrt{9+\sqrt{x}})dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx,\ x > 0$ and $y(0) = \sqrt{7},$ then $y(256) =$ ______.

(A) 3 (B) 9

(C) 16 (D) 80

Solution

(A) It is given that

$$8\sqrt{x}(\sqrt{9+\sqrt{x}})dy = (\sqrt{4+\sqrt{9+\sqrt{x}}})^{-1}dx \qquad \text{(where } x > 0\text{)}$$

Therefore,

$$\frac{dy}{dx} = \frac{(\sqrt{4+\sqrt{9+\sqrt{x}}})^{-1}}{8\sqrt{x}\sqrt{9+\sqrt{x}}} = \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{8\sqrt{x}\sqrt{9+\sqrt{x}}}$$

$$\Rightarrow dy = \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{8\sqrt{x}\sqrt{9+\sqrt{x}}} dx \qquad (1)$$

Integrating Eq. (1), we get

$$\int dy = \int \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{8\sqrt{x}} \frac{1}{\sqrt{9+\sqrt{x}}} dx$$

$$\Rightarrow y = \frac{1}{8}\int \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{9+\sqrt{x}}} dx$$

Let $\sqrt{9+\sqrt{x}} = t$. Differentiating this equation, we get

$$2\frac{1}{\sqrt{9+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} dx = dt \Rightarrow \frac{dx}{\sqrt{x}\sqrt{9+\sqrt{x}}} = 4dt$$

$$\Rightarrow\ y = \frac{1}{8}\int \frac{1}{\sqrt{4+t}} 4dt = \frac{1}{2}\int \frac{1}{\sqrt{4+t}} dt$$

$$\Rightarrow\ y = \frac{1}{2}\frac{(4+t)^{1/2}}{1/2} + C = \sqrt{4+t} + C$$

Substituting $t=\sqrt{9+\sqrt{x}}$, we get

$$y=\sqrt{4+\sqrt{9+\sqrt{x}}}+C$$

It is also given that $y(0)=\sqrt{7}$.

$$\sqrt{7}=\sqrt{4+\sqrt{9+\sqrt{0}}}+C$$
$$\Rightarrow \sqrt{7}=\sqrt{4+\sqrt{9}}+C=\sqrt{4+3}+C$$
$$\Rightarrow \sqrt{7}=\sqrt{7}+C \Rightarrow C=0$$

Therefore, $$y=\sqrt{4+\sqrt{9+\sqrt{x}}}$$

$$\Rightarrow y(256)=\sqrt{4+\sqrt{9+\sqrt{256}}}=\sqrt{4+\sqrt{9+16}}=\sqrt{4+\sqrt{25}}=\sqrt{4+5}$$
$$\Rightarrow y(256)=\sqrt{9}=3$$

4. If f: $\mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that $f''(x)>0$ for all $x \in \mathbb{R}$, and $f\left(\frac{1}{2}\right)=\frac{1}{2}$, $f(1)=1$, then

(A) $f'(1) \le 0$ (B) $0 < f'(1) \le \frac{1}{2}$

(C) $\frac{1}{2} < f'(1) \le 1$ (D) $f'(1) > 1$

Solution

(D) It is given that

$$f''(x)>0 \text{ and } f\left(\frac{1}{2}\right)=\frac{1}{2},\ f(1)=1$$

Using Lagrange's mean value theorem, let us consider that $f:\left[\frac{1}{2},1\right]\to\mathbb{R}$ be continuous and differentiable on $\left(\frac{1}{2},1\right)$, then there exists $c\in\left(\frac{1}{2},1\right)$ such that

$$f'(c)=\frac{f(1)-f\left(\frac{1}{2}\right)}{1-\frac{1}{2}}=\frac{1-\frac{1}{2}}{\frac{1}{2}}=1 \Rightarrow f'(c)=1$$

Since $f'(x)$ is increasing function for $x\in\mathbb{R}$, we get $f'(1)>1$.

5. How many 3 × 3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries of M^TM is 5?

(A) 126 (B) 198

(C) 162 (D) 135

Solution

(B) Let us consider a 3 × 3 matrix

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Therefore, $$M^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

It is given that sum of diagonal of $M^T M$ is 5. Therefore,

$$M^T M = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1^2 + b_1^2 + c_1^2 & a_1a_2 + b_1b_2 + c_2 & a_1a_3 + b_1b_3 + c_1c_3 \\ a_2a_1 + b_2b_1 + c_2c_1 & a_2^2 + b_2^2 + c_2^2 & a_2a_3 + b_2b_3 + c_2c_3 \\ a_3a_1 + b_3b_1 + c_3c_1 & a_3a_2 + b_3b_2 + c_3c_2 & a_3^2 + b_3^2 + c_3^2 \end{bmatrix}$$

$$\Rightarrow (a_1^2 + b_1^2 + c_1^2) + (a_2^2 + b_2^2 + c_2^2) + (a_3^2 + b_3^2 + c_3^2) = 5$$

There are two possible cases:

(i) $0^2 + 0^2 + 0^2 + 0^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 = 5$

This has ${}^9C_5$ combinations possible.

(ii) $1^2 + 2^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 = 5$

This has ${}^9C_7 \times {}^2C_1$ possible combinations.

Therefore, $${}^9C_5 + {}^9C_7 \times {}^2C_1 = \frac{9!}{5! \times 4!} + \frac{9!}{7!2!}\frac{2!}{1!1!}$$

$$\Rightarrow \frac{9 \times 8 \times 7 \times 6 \times 5!}{5! \times 4 \times 3 \times 2 \times 1} + \frac{9 \times 8 \times 7!}{7! \times 2 \times 1} \times 2 \Rightarrow 126 + 72 = 198$$

Thus, the total number of 3×3 matrices is 198.

6. Let $S = \{1, 2, 3, \ldots, 9\}$. For $k = 1, 2, \ldots, 5$, let N_K be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$ ______.

(A) 210 (B) 252

(C) 125 (D) 126

Solution

(D) If N_k be number of subjects of S containing 5 elements each out of which exactly k are odd elements. Therefore,

$$N_1 = {}^5C_1 \times {}^4C_4 = \frac{5!}{4! \times 1!} \times \frac{4!}{4! \times 0!} = 5$$

$$N_2 = {}^5C_2 \times {}^4C_3 = \frac{5!}{3! \times 2!} \times \frac{4!}{3! \times 1!} = \frac{5 \times 4 \times 4}{2} = 40$$

$$N_3 = {}^5C_3 \times {}^4C_2 = \frac{5!}{3! \times 2!} \times \frac{4!}{2! \times 2!} = \frac{5 \times 4}{2} \times \frac{4 \times 3}{2} = 60$$

$$N_4 = {}^5C_4 \times {}^4C_1 = \frac{5!}{4! \times 1!} \times \frac{4!}{3! \times 1!} = 5 \times 4 = 20$$

$$N_5 = {}^5C_5 \times {}^4C_0 = \frac{5!}{5! \times 0!} \times \frac{4!}{4! \times 0!} = 1$$

Hence, $N_1 + N_2 + N_3 + N_4 + N_5 = 5 + 40 + 60 + 20 + 1 = 126$.

7. Three randomly chosen nonnegative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is

(A) $\frac{36}{55}$ (B) $\frac{6}{11}$

(C) $\frac{1}{2}$ (D) $\frac{5}{11}$

Solution

(B) It is given that the integers *x*, *y* and *z* satisfy

$$x + y + z = 10$$

The total number of non-negative integers satisfying this equation is

$$^{10+3-1}C_{3-1} = {}^{12}C_2 = \frac{12!}{10!\times 2!} = \frac{12\times 11}{2} = 66$$

Suppose *z* is even, let $z = 2k$. Therefore,

$$x + y + 2k = 10$$

$$\Rightarrow x + y = 10 - 2k$$

Then, the total number of non-negative solutions is

$$11 + 9 + 7 + 5 + 3 + 1 = 36$$

$$\left(\begin{array}{l}\text{since for } k = 0, 1, 2, 3, 4, 5, \text{ we have} \\ ^{10-2k+2-1}C_{2-1} = {}^{11-2k}C_1 \text{ solutions}\end{array}\right)$$

Therefore, the probability that the integer *z* should be even is

$$\frac{36}{66} = \frac{6}{11}$$

H2>One or More Than One Option Correct Type

This section contains 7 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D) out of which **ONE OR MORE** is(are) correct.

8. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)dt,$ then

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

(C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

Solution

(*) It is given that

$$g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\, dt$$

Differentiating this equation, we get

$$g'(x) = [\sin^{-1}(\sin 2x)]2\cos 2x - [\sin^{-1}(\sin x)]\cos x$$

Therefore, $$g'\left(\frac{\pi}{2}\right) = [\sin^{-1}(\sin \pi)]2\cos \pi - \left[\sin^{-1}\left(\sin \frac{\pi}{2}\right)\right]\cos \frac{\pi}{2}$$

$$\Rightarrow g'\left(\frac{\pi}{2}\right) = 0$$

and $$g'\left(-\frac{\pi}{2}\right)=[\sin^{-1}(-\sin\pi)]2\cos\pi-\left[\sin^{-1}\left(-\sin\frac{\pi}{2}\right)\cos\frac{\pi}{2}\right]=0$$

*Conflict question. None of the options is correct.

9. Let α and β be non-zero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then, which of the following is/are true?

(A) $\tan\left(\frac{\alpha}{2}\right)+\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(B) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=0$

(C) $\tan\left(\frac{\alpha}{2}\right)-\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$

Solution

(A), (C) It is given that

$$2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$$

Using $$\cos\alpha=\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

and $$\cos\beta=\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}$$

we get $$2\left(\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}-\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\right)+\left[\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\times\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}\right]=1$$

$$\Rightarrow 2\left[\frac{\left(1-\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\alpha}{2}\right)-\left(1+\tan^2\frac{\beta}{2}\right)\left(1-\tan^2\frac{\alpha}{2}\right)}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right]$$

$$+\left[\frac{\left(1-\tan^2\frac{\alpha}{2}\right)\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right]=1$$

$$\Rightarrow 2\left(\frac{1-\tan^2\frac{\beta}{2}+\tan^2\frac{\alpha}{2}-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}-1+\tan^2\frac{\alpha}{2}-\tan^2\frac{\beta}{2}+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right)$$

$$+\left[\frac{1-\tan^2\frac{\alpha}{2}-\tan^2\frac{\beta}{2}+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right]=1$$

$$\Rightarrow \begin{array}{l} 2\left[\dfrac{\left(2\tan^2\frac{\alpha}{2}\right)-\left(2\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right] \\ +\left[\dfrac{\left(1-\tan^2\frac{\alpha}{2}\right)-\left(\tan^2\frac{\beta}{2}\right)+\left(\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}\right]=1 \end{array}$$

$$\Rightarrow \left(4\tan^2\frac{\alpha}{2}-4\tan^2\frac{\beta}{2}+1-\tan^2\frac{\alpha}{2}-\tan^2\frac{\beta}{2}+\tan^3\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)=\left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)$$

$$\Rightarrow \left(3\tan^2\frac{\alpha}{2}-5\tan^2\frac{\beta}{2}+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}+1\right)=\left(1+\tan^2\frac{\alpha}{2}+\tan^2\frac{\beta}{2}+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)$$

$$\Rightarrow \left(2\tan^2\frac{\alpha}{2}-6\tan^2\frac{\beta}{2}\right)=0$$

$$\Rightarrow \left(\tan^2\frac{\alpha}{2}\right)=\left(3\tan^2\frac{\beta}{2}\right)$$

$$\Rightarrow \left(\tan\frac{\alpha}{2}\right)=\pm\left(\sqrt{3}\tan\frac{\beta}{2}\right)$$

$$\Rightarrow \left(\tan\frac{\alpha}{2}\right)\pm\left(\sqrt{3}\tan\frac{\beta}{2}\right)=0$$

Hence, options (A) and (C) are correct.

10. If *f*: $\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f'(x)>2f(x)$ for all $x\in\mathbb{R}$, and $f(0)=1$, then

(A) $f(x)$ is increasing in $(0,\infty)$.

(B) $f(x)$ is decreasing in $(0,\infty)$.

(C) $f(x)>e^{2x}$ in $(0,\infty)$.

(D) $f'(x)<e^{2x}$ in $f'(x)<e^{2x}$.

Solution

(A), (C) It is given that *f*: $\mathbb{R}\to\mathbb{R}$ is differentiable function such that $f'(x)>2f(x)$ for all $x\in\mathbb{R}$ and $f(0)=1$.

$$f'(x)>2f(x)\Rightarrow f'(x)-2f(x)>0$$

Multiplying this with e^{-2x}, we get

$$e^{-2x}f'(x)-2e^{-2x}f(x)>0\Rightarrow\frac{d}{dx}[e^{-2x}f(x)]>0$$

Therefore, $e^{-2x}f(x)$ is an increasing function.

Let $e^{-2x}f(x)=g(x)$.

- For $x = 0$: $e^{-20} f(0) = 1 = g(0)$.
- for $x > 0$: $g(x) > g(0)$.

That is, $$e^{-2x} f(x) > f(0) \Rightarrow e^{-2x} f(x) > 1. \Rightarrow f(x) > \frac{1}{e^{-2x}}$$

Thus, $$f(x) > e^{2x} \qquad [\text{in } (0, \infty]$$

(1)

Hence, option (C) is correct.

It is given that $f'(x) > 2f(x)$. Now, using (1), we get

$$f'(x) > 2f(x) > 2e^{2x}$$

Thus, $f(x)$ is an increasing function [in $(0, \infty$].

Hence, option (A) is correct.

11. Let $f(x) = \frac{1 - x(1 + |1 - x|)}{|1 - x|} \cos\left(\frac{1}{1 - x}\right)$ for $x \neq 1$. Then

(A) $\lim_{x \to 1^{-1}} f(x) = 0$.

(B) $\lim_{x \to 1^{-1}} f(x)$ does not exist.

(C) $\lim_{x \to 1^{-1}} f(x) = 0$.

(D) $\lim_{x \to 1^{+1}} f(x)$ does not exist.

Solution

(A), (D) It is given that

$$f(x) = \frac{1 - x(1 + |1 - x|)}{|1 - x|} \cos\left(\frac{1}{1 - x}\right) \qquad (\text{for } x \neq 1)$$

Then, $$\lim_{x \to 1^+} \frac{1 - x(1 + |1 - x|)}{|1 - x|} \cos\left(\frac{1}{1 - x}\right) = \lim_{x \to 1^+} \frac{1 - x(1 + x - 1)}{(x - 1)} \cos\left(\frac{1}{1 - x}\right)$$

$$= \lim_{x \to 1^+} \frac{1 - x^2}{(x - 1)} \cos\left(\frac{1}{1 - x}\right) = \lim_{x \to 1^+} \frac{(1 - x)(1 + x)}{-(1 - x)} \cos\left(\frac{1}{1 - x}\right)$$

$$= \lim_{x \to 1^+} -(1 + x) \cos\left(\frac{1}{1 - x}\right)$$

Thus, the limit does not exist.

Hence, option (D) is correct.

$$\lim_{x \to 1^-} \frac{1 - x(1 + |1 - x|)}{|1 - x|} \cos\left(\frac{1}{1 - x}\right) = \lim_{x \to 1^-} \frac{1 - x(1 + 1 - x)}{1 - x} \cos\left(\frac{1}{1 - x}\right)$$

$$= \lim_{x \to 1^-} \frac{1 - x(2 - x)}{1 - x} \cos\left(\frac{1}{1 - x}\right)$$

$$= \lim_{x \to 1^-} \frac{1 - 2x + x^2}{1 - x} \cos\left(\frac{1}{1 - x}\right) = \lim_{x \to 1^-} \frac{(x - 1)^2}{(1 - x)} \cos\left(\frac{1}{1 - x}\right)$$

$$= \lim_{x\to 1^-}(1-x)\cos\left(\frac{1}{1-x}\right) = 0$$

Hence, option (A) is correct.

12. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then

(A) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$.

(B) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$.

(C) $f(x)$ attains its maximum at $x = 0$.

(D) $f(x)$ attains its minimum at $x = 0$.

Solution

(B), (C) It is given that

$$f(x) = \begin{vmatrix} \cos 2x & \cos 2x & \sin 2x \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$$

$$\Rightarrow f(x) = \cos 2x(\cos^2 x + \sin^2 x) - \cos 2x(-\cos^2 x + \sin^2 x) + \sin 2x(-\sin x\cos x - \sin x\cos x)$$

Using $\sin^2 x + \cos^2 x = 1$, we get

$$f(x) = \cos 2x - \cos 2x(-\cos^2 x + \sin^2 x) + \sin 2x(-2\sin x\cos x)$$

$$= \cos 2x - \cos 2x(-\cos 2x) + \sin 2x(-\sin 2x) = \cos 2x + \cos^2 2x - \sin^2 2x$$

Using $\cos^2 x + \sin^2 x = 1$ and $\sin^2 2x = 1 - \cos^2 2x$, we get

$$f(x) = \cos 2x + \cos^2 2x - (1 - \cos^2 2x)$$

$$= \cos 2x + \cos^2 2x - 1 + \cos^2 2x$$

Therefore, $f(x) = 2\cos^2 2x + \cos 2x - 1 = \cos 4x + \cos 2x$

Differentiating this, we get

$$f'(x) = 4(-\sin 4x) + 2(-\sin 2x)$$

$$f'(x) = -4\sin 4x - 2\sin 2x$$

Now $f'(x) = 0$ gives

$$-4\sin 4x - 2\sin 2x = 0$$

Using $\sin 2x = 2\sin x\cos x$, we get

$$-4(2\sin 2x\cos 2x) - 2\sin 2x = 0$$

$$-8\sin 2x\cos 2x - 2\sin 2x = 0$$

$$-2\sin 2x(4\cos 2x + 1) = 0$$

$$\Rightarrow 2\sin 2x = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = 0, \pi, -\pi \Rightarrow x = 0, \frac{\pi}{2}, \frac{-\pi}{2}$$

and $4\cos 2x + 1 = 0 \Rightarrow \cos 2x = \frac{-1}{4} \Rightarrow 2x = \cos^{-1}\left(\frac{-1}{4}\right) \Rightarrow 2x = 1.8 + 2\pi n$

where $n = ..., -2, -1, 0, 1, 2, \ldots$, which gives 4 points in the range $(-\pi, \pi)$.

Thus, the total points in $(-\pi, \pi)$ range are 7.

Hence, option (B) is correct.

Now, $$f'(x) = [-4(\cos 4x)4] - [2\cos(2x)2] = -16\cos 4x - 4\cos 2x$$

At $x = 0$, we get

$$f''(x) = -16 - 4 = -20$$

(maxima)

Thus, at $x = 0$, the function $f(x)$ attains maximum at $x = 0$.

Hence, option (C) is correct.

13. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in \mathbb{R}^2 : x^3 \le y \le x, 0 \le x \le 1\}$ into two equal parts, then

(A) $0 < \alpha \le \frac{1}{2}$ (B) $\frac{1}{2} < \alpha < 1$

(C) $2\alpha^4 - 4\alpha^2 + 1 = 0$ (D) $\alpha^4 + 4\alpha^2 - 1 = 0$

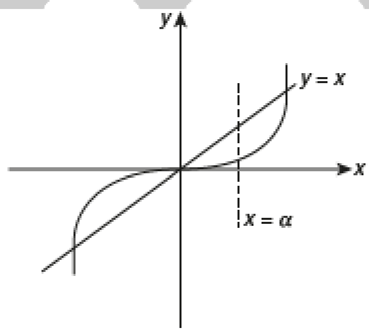
Solution

(B), (C) Let us consider $y = x^3$ and $y = x$.

Then, the area between these two curves in region $0 \le x \le 1$ is

$$A = \int_0^1 (x - x^3)dx$$

It is given that the line $x = \alpha$ divides the area under the curve into two equal parts. Therefore,



$$\int_0^\alpha (x - x^3)dx = \int_\alpha^1 (x - x^3)dx$$

$$\Rightarrow \left.\frac{x^2}{2} - \frac{x^4}{4}\right|_0^\alpha = \left.\frac{x^2}{2} - \frac{x^4}{4}\right|_\alpha^1$$

$$\Rightarrow \left(\frac{\alpha^2}{2} - 0\right) - \left(\frac{\alpha^4}{4} - 0\right) = \left(\frac{1}{2} - \frac{\alpha^2}{2}\right) - \left(\frac{1}{4} - \frac{\alpha^4}{4}\right)$$

$$\Rightarrow \frac{\alpha^2}{2} - \frac{\alpha^4}{4} = \frac{1}{2} - \frac{\alpha^2}{2} - \frac{1}{4} + \frac{\alpha^4}{4}$$

$$\Rightarrow \frac{\alpha^2}{2} - \frac{\alpha^4}{4} - \frac{1}{2} + \frac{\alpha^2}{2} + \frac{1}{4} - \frac{\alpha^4}{4} = 0$$

$$\Rightarrow \alpha^2 - \frac{\alpha^4}{2} - \frac{1}{4} = 0$$

$$\Rightarrow \frac{4\alpha^2 - 2\alpha^4 - 1}{4} = 0 \Rightarrow 4\alpha^2 - 2\alpha^4 - 1 = 0$$

$$\Rightarrow -(2\alpha^4 - 4\alpha^2 + 1) = 0 \Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0$$

Hence, option (C) is correct.

Now, let us consider the equation $2\alpha^4 - 4\alpha^2 + 1 = 0$.

Let $\alpha^2 = u$. Therefore,

$$2u^2 - 4u + 1 = 0$$

$$\Rightarrow u = \frac{4 \pm \sqrt{16 - 4\times 2}}{2\times 2}$$

$$\left(\begin{array}{l}\text{since for equation } ax^2 + bx + c = 0, \\ \text{the discriminant} = b^2 - 4ac = D \text{ and} \\ \text{the roots of } x \text{ are } \dfrac{-b \pm \sqrt{D}}{2a}\end{array}\right)$$

$$\Rightarrow u = \frac{4 \pm \sqrt{16-8}}{4} = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm \sqrt{4\times 2}}{4}$$

$$\Rightarrow u = \frac{4 \pm 2\sqrt{2}}{4} = 1 \pm \frac{\sqrt{2}}{2} \Rightarrow u = 1 \pm \frac{1}{\sqrt{2}}$$

Substituting $u = \alpha^2$, we get

$$\alpha^2 = 1 \pm \frac{1}{\sqrt{2}}$$

From $\alpha^2 = 1 \pm \frac{1}{\sqrt{2}}$, we get $\frac{1}{2} < \alpha < 1$

Thus, option (B) is correct.

14. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, then

(A) $I > \log_e 99$ (B) $I < \log_e 99$

(C) $I < \frac{49}{50}$ (D) $I > \frac{49}{50}$

Solution

(B), (D)

It is given that

$$I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$$

$$= \sum_{k=1}^{98} (k+1) \int_k^{k+1} \frac{1}{x(x+1)} dx = \sum_{k=1}^{98} (k+1) \int_k^{k+1} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \sum_{k=1}^{98} (k+1) \left[\ln x - \ln(x+1)\Big|_k^{k+1}\right]$$

$$= \sum_{k=1}^{98} (k+1)\left[(\ln(k+1) - \ln k) - (\ln(k+1+1) - \ln(k+1)\right]$$

$$= \sum_{k=1}^{98} (k+1)\left[\ln(k+1) - \ln k - \ln(k+2) + \ln(k+1)\right]$$

$$= \sum_{k=1}^{98} (k+1)\ln(k+1) - (k+1)\ln k - (k+1)\ln(k+2) + (k+1)\ln(k+1)$$

$$= \sum_{k=1}^{98} (k+1)\ln(k+1) - k\ln k - \ln k - (k+1)\ln(k+2) + k\ln(k+1) + \ln(k+1)$$

Rearranging the terms, we get

$$I = \sum_{k=1}^{98} (k+1)\ln(k+1) - k\ln k - \sum_{k=1}^{98} (k+1)\ln(k+2) - k\ln(k+1) + \sum_{k=1}^{98} \ln(k+1) - \ln k$$

That is, $I = 99 \ln 99 - 99 \ln 100 + \ln 2 + \ln 99 = \ln 99^{99} - \ln 100^{99} + \ln 2 + \ln 99$

$$I = \ln\left(\frac{99^{99}}{100^{99}} \times 2 \times 99\right) = \ln\left(\frac{99^{100} \times 2}{100^{99}}\right) \qquad (1)$$

Now, let us consider that 100^{99} is written as

$$100^{99} = (99 + 1)^{99}$$

Using binomial expansion, we get

$$(100)^{99} = {}^{99}C_0 + {}^{99}C_1(99)^1 + {}^{99}C_2(99)^2 + \ldots + {}^{99}C_{98}(99)^{98} + {}^{99}C_{99}(99)^{99}$$

$$= {}^{99}C_0 + {}^{99}C_1(99)^1 + {}^{99}C_2(99)^2 + \ldots + (99)^{99} + (99)^{99}$$

$$\Rightarrow (100)^{99} > 2\cdot(99)^{99} \qquad \text{(considering last two terms)}$$

$$\Rightarrow \frac{2\cdot(99)^{99}}{(100)^{99}} < 1$$

Multiplying both sides by 99, we get

$$\frac{2\cdot(99)^{100}}{(100)^{99}} < 99$$

Taking natural logarithm on both sides, we get

$$\ln\frac{2(99)^{100}}{(100)^{99}} < \ln 99 \Rightarrow I < \ln 99 \qquad (\ln \equiv \log_e)$$

Hence, option (B) is correct.

Now, we know that

$$I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$$

$$\sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx > \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{(x+1)^2} dx$$

$$\Rightarrow I > \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{(x+1)^2} dx = \sum_{k=1}^{98} (k+1) \int_k^{k+1} (x+1)^{-2} dx$$

$$\Rightarrow \ I > \sum_{k=1}^{98}(k+1)\left(\left.\frac{(x+1)^{-2+1}}{-2+1}\right|_{k}^{k+1}\right) = \sum_{k=1}^{98}(k+1)\left(\left.\frac{-1}{(x+1)}\right|_{k}^{k+1}\right)$$

$$\Rightarrow$$

$$I > \sum_{k=1}^{98}(k+1)\left(\frac{-1}{(k+1+1)}+\frac{1}{(k+1)}\right) = \sum_{k=1}^{98}(k+1)\left(\frac{1}{(k+1)}-\frac{1}{(k+2)}\right)$$

$$\Rightarrow \ I > \sum_{k=1}^{98}(k+1)\left(\frac{k+2-k-1}{(k+1)(k+2)}\right) = \sum_{k=1}^{98}(k+1)\frac{1}{(k+1)(k+2)}$$

$$\Rightarrow \ I > \sum_{k=1}^{98}\frac{1}{(k+2)} = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{100}$$

These are total 98 terms.

Now, $\frac{1}{100}+\frac{1}{100}+\cdots+\frac{1}{100}$ (98 terms) $=\frac{98}{100} < \frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{100}$

Therefore, $I > \frac{98}{100} = \frac{49}{50} \Rightarrow I > \frac{49}{50}$

Hence, option (D) is correct

<H1>Paragraph Type

This section contains 2 paragraphs. Based on each paragraph, there are 2 questions. Each question has 4 options (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

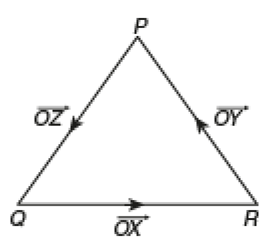
Paragraph for Questions 15 and 16: Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, respectively, of a triangle PQR.

15. $|\overrightarrow{OX} \times \overrightarrow{OY}| =$ _____.

(A) $\sin(P + Q)$ (B) $\sin 2R$
(C) $\sin(P + R)$ (D) $\sin(Q + R)$

Solution

(A) The given geometrical situation is depicted in the following figure:



Now, $\overrightarrow{OX} = \frac{\overrightarrow{QR}}{QR}$

and $\overrightarrow{OY} = \frac{\overrightarrow{RP}}{RP}$

Therefore, $(\overrightarrow{OX} \times \overrightarrow{OY}) = \frac{\overrightarrow{QR}}{QR} \times \frac{\overrightarrow{RP}}{RP} = \frac{\overrightarrow{QR} \times \overrightarrow{RP}}{PQ}$

$$= \frac{PQ \sin R}{PQ} = \sin R = \sin(\pi - (P + Q) = \sin(P + Q)$$

16. If the triangle PQR varies, then the minimum value of $\cos(P + Q) + \cos(Q + R) + \cos(R + P)$ is

(A) $-\frac{5}{3}$ (B) $-\frac{3}{2}$

(C) $\frac{3}{2}$ (D) $\frac{5}{3}$

Solution

(B) It is given that

$$\cos(P + Q) + \cos(Q + R) + \cos(R + P) = \cos R + \cos P + \cos Q$$

In the given triangle, the maximum value of

$$\cos P + \cos Q + \cos R = \frac{3}{2}$$

Therefore, the minimum value of

$$\cos P + \cos Q + \cos R = \frac{-3}{2}$$

$$\Rightarrow \cos(P + Q) + \cos(Q + R) + \cos(R + P) = \frac{-3}{2}$$

Paragraph for Questions 17 and 18: Let p, q be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0, 1, 2, \ldots$, let $a_n = p\alpha^n + q\beta^n$.

FACT: If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$.

17. a_{12} = _____.

(A) $a_{11} - a_{10}$ (B) $a_{11} + a_{10}$

(C) $2a_{11} + a_{10}$ (D) $a_{11} + 2a_{10}$

Solution

(B) It is given that $x^2 - x - 1 = 0$.

Also, α and β are roots of equation and $\alpha \neq \beta$.

Let p and q be integers and $p\alpha^n + q\beta^n = a_n$.

Since α and β are the roots of $x^2 = x + 1$, we get

$$\alpha^2 = \alpha + 1 \text{ and } \beta^2 = \beta + 1$$

Therefore,

$$\begin{aligned} a_{11} + a_{10} &= p\alpha^{11} + q\beta^{11} + p\alpha^{10} + q\beta^{10} \\ &= p\alpha^{11} + p\alpha^{10} + q\beta^{11} + q\beta^{10} \\ &= p\alpha^{10}(\alpha + 1) + q\beta^{10}(\beta + 1) \\ &= p\alpha^{10}\alpha^2 + q\beta^{10}\beta^2 \\ &= p\alpha^{12} + q\beta^{12} = a_{12} \end{aligned}$$

That is, $a_{11} + a_{10} = a_{12}$.

54. If $a_4 = 28$, then $p + 2q$ = _____.

(A) 21 (B) 14

(C) 7 (D) 12

Solution

(D) It is given that $a_4 = 28$. Using $a_n = p\alpha^n + q\beta^n$, we get

$$a_n - a_{n-1} = p\alpha^n + q\beta^n - (p\alpha^{n-1} + q\beta^{n-1})$$
$$= p\alpha^n - p\alpha^{n-1} + q\beta^n - q\beta^{n-1}$$
$$= p\alpha^{n-2}(\alpha^2 - \alpha) + q\beta^{n-2}(\beta^2 - \beta)$$
$$= p\alpha^{n-2} + q\beta^{n-2}$$

Therefore, $a_n - a_{n-1} = a_{n-2}$.

$$\Rightarrow a_n = a_{n-1} + a_{n-2}$$
$$\Rightarrow a_4 = a_3 + a_2 = (a_2 + a_1) + (a_1 + a_0)$$
$$= a_2 + 2a_1 + a_0$$
$$\Rightarrow a_4 = (a_1 + a_0) + 2a_1 + a_0 = 2a_0 + 3a_1$$
$$\Rightarrow a_4 = 2(p\alpha^\circ + q\beta^\circ) + 3(p\alpha^1 + q\beta^1)$$
$$\Rightarrow a_4 = 2p + 2q + 3(p\alpha + q\beta)$$

Now, from $x^2 - x - 1 = 0$, the roots of the equation are

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$
$$\Rightarrow \alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}$$
$$\Rightarrow a_4 = 2p + 2q + 3\left(p\left(\frac{1+\sqrt{5}}{2}\right) + q\left(\frac{1-\sqrt{5}}{2}\right)\right)$$
$$= 2p + 2q + \frac{3}{2}p + \frac{3}{2}p\sqrt{5} + \frac{3}{2}q - \frac{3}{2}q\sqrt{5}$$
$$= 2p + 2q + p\left(\frac{3}{2} + \frac{3}{2}\sqrt{5}\right) + q\left(\frac{3}{2} - \frac{3}{2}\sqrt{5}\right)$$

It is given that if a and b are rational numbers and $a + b\sqrt{5} = 0$; then, $a = 0 = 6$. Therefore,

$$a_4 = 2p + 2q + \frac{3}{2}p + \frac{3}{2}q$$

and $$\frac{3}{2}p = \frac{3}{2}q \Rightarrow p = q$$

$$\Rightarrow a_4 = 2p + 2p + \frac{3}{2}p + \frac{3}{2}p = 7p$$
$$\Rightarrow 28 = 7p \Rightarrow p = 4 \Rightarrow q = 4$$
$$\Rightarrow p + 2q = 4 + 2 \times 4 = 12$$