MATHEMATICS (CBSE 2013)

Time Allowed: 3 hours **Maximum marks: 100**

General Instructions

- All questions are compulsory.
- The question paper consists of 29 questions divided into three sections A, B and C. Section A comprises of **10** questions of **one mark** each, Section B comprises of **12** questions of **four marks** each and Section C comprises of **7** questions of **six marks** each.
- All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
- Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Section A

Question numbers 1 to 10 carry 1 mark each.

1. Write the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant.

Solution

The given equation is

$$y = mx \qquad (1)$$

Differentiating Eq. (1) with respect to x, we get

$$\frac{d}{dx}(y) = \frac{d}{dx}(mx)$$

$$\frac{dy}{dx} = m\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = m. \qquad (2)$$

Substituting Eq. (2) in Eq. (1), we get

$$y = x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

which the required differential equation.

2. If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32} \cdot A_{32}$.

Solution

Let $B = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$. Then,

$$a_{32} \cdot A_{32} = -5\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$$

$$= -5(8-30) = -5(-22) = 110.$$

3. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

Solution

$$\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]=\tan^{-1}\left[2\sin\left(\cancel{2}\cdot\frac{\pi}{\cancel{6}_3}\right)\right]$$
$$=\tan^{-1}\left[2\sin\left(\frac{\pi}{3}\right)\right]$$
$$=\tan^{-1}\left[2\frac{\sqrt{3}}{2}\right]=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}.$$

4. For what value of x, is the matrix $A=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix}$ a skew-symmetric matrix?

Solution

For A to be skew symmetric matrix

$$A'=-A \qquad (1)$$

That is,

$$A'=\begin{bmatrix}0 & -1 & x\\ 1 & 0 & -3\\ -2 & 3 & 0\end{bmatrix} \qquad (2)$$

where

$$A=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix} \qquad (3)$$

Substituting Eqs. (2) and (3) in Eq. (1), we get

$$\begin{bmatrix}0 & -1 & x\\ 1 & 0 & -3\\ -2 & 3 & 0\end{bmatrix}=-\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix},$$

which implies that $x = +2$.

5. Write the principal value of $\tan^{-1}(\sqrt{3})-\cot^{-1}(-\sqrt{3})$.

Solution

The principal value of $\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$ is obtained as follows:

$$\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})=\frac{\pi}{3}-\left(\pi-\frac{\pi}{6}\right)$$
$$=\frac{\pi}{3}-\frac{5\pi}{6}=\frac{2\pi-5\pi}{6}=\frac{-\cancel{3}\pi}{\cancel{6}_2}=\frac{-\pi}{2}.$$

6. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$, and write which value does the question indicate.

Solution

The total revenue from the sale of x units of a product is given by

$$R(x)=3x^2+36x+5.$$

The marginal revenue is

$$\frac{d}{dx}R(x) = \frac{d}{dx}(3x^2 + 36x + 5)$$
$$= 6x + 36.$$

Therefore, the marginal revenue when $x = 5$ is obtained as $(6\times 5)+36 = 66.$ The question indicated that more revenue implies more money for the welfare of employees.

7. Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$.
Solution
The length of perpendicular drawn from a given point on the plane, $2x - 3y + 6z + 21 = 0$, is calculated as follows:

$$P = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$
$$= \left|\frac{2(0) - 3(0) + 6(0) + 21}{\sqrt{4 + 9 + 36}}\right| = \left|\frac{21}{\sqrt{49}}\right| = \frac{21}{7} = 3\text{units.}$$

8. Find $|\vec{x}|$, if for a unit vector $\vec{a}, (\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 15$.
Solution

$$(\vec{x} - \vec{a})\cdot(\vec{x} + \vec{a}) = 15$$
$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 15$$
$$\Rightarrow |\vec{x}|^2 - 1 = 15 \qquad (\because |\vec{a}| = 1)$$
$$\Rightarrow |\vec{x}|^2 = 16$$
$$\Rightarrow |\vec{x}| = 4.$$

9. If matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = pA$, then write the value of p.

Solution
We have

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \text{ and } A^2 = pA$$

That is,

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = p\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = \begin{bmatrix} 2P & -2P \\ -2P & 2P \end{bmatrix}$$

Using equality of two matrices, we get

$$2P = 8 \Rightarrow P = 4.$$

10. A and B are two points with position vectors $2\vec{a} - 3\vec{b}$ and $6\vec{b} - \vec{a}$, respectively. Write the position vector of a point P which divides the line segment AB internally in the ratio 1 : 2.
Solution
Position vector of point A is $2\vec{a} - 3\vec{b}$ and the position vector of point B is $6\vec{b} - \vec{a}$. The position vector of a point P which divides the line segment AB internally in the ratio 1 : 2 is

$$P = \frac{2(2\vec{a} - 3\vec{b}) + (6\vec{b} - \vec{a})}{2+1}$$
$$= \frac{4\vec{a} - 6\vec{b} + 6\vec{b} - \vec{a}}{3}$$
$$= \frac{3\vec{a}}{3} = \vec{a}.$$

SECTION B

Question numbers 11 to 22 carry 4 marks each.

11. Differentiate the following with respect to x :

$$\sin^{-1}\left(\frac{2^{x+1} \cdot 3^x}{1+(36)^x}\right)$$

Solution

Let $y = \sin^{-1}\left(\frac{2^{x+1} \cdot 3^x}{1+(36)^x}\right)$. Therefore,

$$y = \sin^{-1}\left(\frac{2^{x+1} \cdot 3^x}{1+(36)^x}\right) = \sin^{-1}\left(\frac{2.2^x.3^x}{1+(6^2)^x}\right)$$
$$= \sin^{-1}\left(\frac{2.6^x}{1+6^{2x}}\right)$$
$$= 2\tan^{-1}(6^x)\left[\sin^{-1}\frac{2x}{1+x^2} = 2\tan^{-1}x\right].$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{2}{1+(6^x)^2} \cdot \frac{d}{dx}6^x$$
$$= \frac{2}{1+6^{2x}} \cdot 6^x \cdot \log 6$$
$$= \frac{2\log 6 \cdot 6^x}{1+6^{2x}}.$$

12. Evaluate:

$$\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx$$

Solution

Let $I = \int \frac{\cos^2 x - \cos^2 x}{\cos x - \cos x} dx$. Therefore,

$$I=\int\frac{\cos^2 x-\cos^2 x}{\cos x-\cos x}dx=\int\frac{(2\cos^2 x-1)-(2\cos^2 x-1)}{(\cos x-\cos x)}dx$$
$$=\int\frac{2\cos^2 x-\cancel{1}-2\cos^2 x+\cancel{1}}{(\cos x-\cos x)}dx$$
$$=\int\frac{2\cos^2 x-2\cos^2 x}{\cos x-\cos x}dx$$
$$=2\int\frac{\cos^2 x-\cos^2 x}{\cos x-\cos x}dx$$
$$=2\int\frac{\cancel{(\cos x-\cos x)}(\cos x+\cos x)}{\cancel{(\cos x-\cos x)}}dx$$
$$=2\int(\cos x+\cos x)dx$$
$$=2(\sin x+x\cos x)+c.$$

OR

Evaluate:

$$\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$$

Solution

Let $I=\int\frac{x+2}{\sqrt{x^2+2x+3}}dx.$ Therefore,

$$I=\int\frac{x+2}{\sqrt{x^2+2x+3}}dx=\int\frac{x+1}{\sqrt{x^2+2x+3}}dx=\int\frac{1}{\sqrt{x^2+2x+3}}dx$$
$$=I_1+I_2$$

In $I_1=\int\frac{x+2}{\sqrt{x^2+2x+3}}$, let us consider that $\sqrt{x^2+2x+3}=t.$ Therefore,

$$\sqrt{x^2+2x+3}=t\Rightarrow\frac{x+1}{\sqrt{x^2+2x+3}}dx=dt.$$

Thus,

$$I_1=\int\frac{dt}{t}=\log t=\log\left|\sqrt{x^2+2x+3}\right|=\log\left|(x+1)+\sqrt{x^2+2x+3}\right|$$

and

$$I_2=\int\frac{1}{\sqrt{x^2+2x+3}}dx=\int\frac{1}{(x+1)^2+(\sqrt{2})^2}dx=\log\left|(x+1)+\sqrt{x^2+2x+3}\right|$$

Therefore,

$$I=I_1+I_2=\log\left|\sqrt{x^2+2x+3}\right|+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+c.$$

13. Using properties of determinants, prove the following:

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}=9y^2(x+y)$$

Solution

$$\text{LHS }=\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

$C_1\rightarrow C_1+C_2+C_3$ gives

$$\begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix} = 3(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$$

$R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$ gives

$$3(x+y)\begin{vmatrix} 0 & -y & 2y \\ 0 & -2y & y \\ 1 & x+2y & x \end{vmatrix} = 3(x+y)(y^2)\begin{vmatrix} 0 & -1 & 2 \\ 0 & -2 & 1 \\ 1 & x+2y & x \end{vmatrix}$$

Expanding along C_1, we get

$$3(x+y)(y^2)\left[1\begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix}\right].$$

14. Show that:

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$$

Solution

Let $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = x$. This implies that

$$\frac{1}{2}\sin^{-1}\frac{3}{4} = \tan^{-1} x$$

$$\Rightarrow \sin^{-1}\frac{3}{4} = 2\tan^{-1} x.$$

$$\Rightarrow \tan^{-1}\frac{3}{\sqrt{7}} = \tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \frac{2x}{1-x^2} = \frac{3}{\sqrt{7}}.$$

Therefore, $3x^2 + 2\sqrt{7}x - 3 = 0$. This implies that

$$x = \frac{-2\sqrt{7} \pm 8}{6}$$

$$= \frac{-2\sqrt{7} \pm 8}{6}$$

$$= \frac{8-2\sqrt{7}}{6} = \frac{4-\sqrt{7}}{3}$$

$$= \frac{4-\sqrt{7}}{3}.$$

OR

Solve the following equation:

$$\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

Solution

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$
$$= \sin\left(\sin^{-1}\frac{4}{5}\right)$$
$$= \frac{4}{5}$$
$$\Rightarrow \tan^{-1}x = \cos^{-1}\left(\frac{4}{5}\right)$$
$$= \tan^{-1}\left(\frac{3}{4}\right)$$
$$\Rightarrow x = \frac{3}{4}.$$

15. Consider $f : \mathbf{R}_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where $\mathbf{R}_+$ is the set of all non-negative real numbers.

Solution

Let $x_1, x_2, \in \mathbf{R}_+$ and $f(x_1) = f(x_2)$. This implies that

$$x_1^2 + 4 = x_2^2 + 4$$
$$\Rightarrow x_1^2 = x_2^2$$
$$\Rightarrow x_1 = x_2.$$

Therefore, f is1−1. Let $y t\{u, x\}$ and $f(x) = y$, which implies that

$$\Rightarrow x^2 + u = y$$
$$\Rightarrow x^2 = y - u$$
$$\Rightarrow x = \sqrt{y-u} \quad \in \mathbf{R}_+ \qquad (\because y \geq u)$$

Therefore, f is onto. Hence, f is bijective and so invertible and $f^{-1}(y) = \sqrt{y-u}$.

16. Find the value of k, for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, \text{ if } -1 \leq x < 0 \\ \dfrac{2x+1}{x-1}, \text{ if } 0 \leq x < 1 \end{cases}$$

is continuous at $x = 0$.

Solution

LHS is given by

$$\lim_{x\to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} \times \frac{\sqrt{1+kx} + \sqrt{1-kx}}{\sqrt{1+kx} + \sqrt{1-kx}} = \lim_{x\to 0^-} \frac{(1+kx)-(1-kx)}{x(\sqrt{1+kx} + \sqrt{1-kx})}$$
$$= \lim_{x\to 0^-} \frac{2K\cancel{x}}{\cancel{x}(\sqrt{1+kx} + \sqrt{1-kx})}$$
$$= \frac{2k}{2} = k.$$

RHS is given by

$$\lim_{x\to 0^+} \frac{2x+1}{x-1} = -1$$
$$f(0) = -1.$$

Therefore, for $f(x)$ to be continuous at $x = 0$, we have

$$\text{LHS} = \text{RHS} = f(0)$$

That is, $k = -1$.

OR

If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$.

Solution

Given that $x = a\cos^3\theta$ and $y = a\sin^3\theta$. When we write the given equations in parametric form, we have

$$\frac{dx}{d\theta} = 3a\cos^2\theta \cdot \frac{d}{d\theta}\cos\theta$$
$$= -3a\cos^2\theta\sin\theta.$$

and

$$\frac{dy}{d\theta} = 3a\sin^2\theta \cdot \frac{d}{d\theta}\sin\theta$$
$$= 3a\sin^2\theta\cos\theta$$

Now

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$
$$= \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$
$$= -\tan\theta.$$

Differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = -\sec^2\theta \cdot \frac{d\theta}{dx}$$
$$= -\sec^2\theta \cdot \left(\frac{-1}{3a\cos^2\theta\sin\theta}\right)$$
$$= \frac{\sec^2\theta}{3a\sin\theta}.$$

At $\theta = \frac{\pi}{6}$, we get

$$\frac{d^2y}{dx^2} = \frac{\sec^4\left(\frac{\pi}{6}\right)}{3a\sin\left(\frac{\pi}{6}\right)} = \frac{\left(\frac{2}{\sqrt{3}}\right)^4}{3a\left(\frac{1}{2}\right)} = \frac{32}{27a}.$$

17. Show that the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$
$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

are intersecting. Hence find their point of intersection.

Solution

We have

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) \qquad (1)$$

and

$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k}) \qquad (2)$$

Any point, say P, on line (1) is

$$(3+\lambda,\ 2+2\lambda,\ -4+2\lambda)$$

and any point, say Q, on line (2) is

$$(5+3\mu,\ -2+2\mu,\ 6\mu)$$

If the lines intersect, points P and Q must coincide for same values of λ and μ. Therefore,

$$3+\lambda = 5+3\mu.$$

That is,

$$\lambda - 3\mu = 2 \qquad (3)$$

Also $2+2\lambda = -2+2\mu$, that is,

$$2\lambda - 2\mu = -4 \qquad (4)$$

and $-4+2\lambda = 6\mu$, that is,

$$2\lambda - 6\mu = 4 \qquad (5)$$

Solving Eqs. (3) and (4), we get λ = –4 and μ = –2. Since these values satisfy Eq. (5), the lines are intersecting and hence the point of intersection is (–1, –6, –12)

OR

Find the vector equation of the plane through the points (2, 1, –1) and (–1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$.

Solution

The equation of plane through (2, 1, –1) is

$$a(x-2) + b(y-1) + c(z+1) = 0 \qquad (1)$$

and (–1, 3, 4) lies on it. Therefore,

$$-3a + 2b + 5c = 0 \qquad (2)$$

As Eq. (1) is perpendicular to $x - 2x + 4z = 10$, we have

$$a - 2b + 4c = 0 \qquad (3)$$

Solving Eqs. (2) and (3) using cross multiplication, we get

$$\frac{a}{\begin{matrix} 2 & & 5 \\ -2 & \times & 4 \end{matrix}} = \frac{b}{\begin{matrix} 5 & & -3 \\ -4 & \times & 1 \end{matrix}} = \frac{c}{\begin{matrix} -3 & & 2 \\ 1 & \times & -2 \end{matrix}}$$

$$\Rightarrow \frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-12}$$

$$\Rightarrow \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda \text{ (say)}$$

Therefore, the equation of plane is

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0$$
$$\Rightarrow 18(x-2)+17(y-1)+4(z+1)=0$$

That is, $18x+17y+4z=49$ and vector equation is

$$\vec{r}\cdot(18\hat{i}+17\hat{j}+4\hat{k})=49.$$

18. The probabilities of two students A and B coming to the school in time are $\frac{3}{7}$ and $\frac{5}{7}$, respectively. Assuming that the events, 'A coming in time' and 'B coming in time' are independent, find the probability of only one of them coming to the school in time. Write at least one advantage of coming to school in time.

Solution

Here $P(A)=\frac{3}{7}$ and $P(B)=\frac{5}{7}$. Also $P(\bar{A})=\frac{4}{7}$ and $P(\bar{B})=\frac{2}{7}$. The probability, P, of only one of them coming in time is given by

$$P(A\,\text{and}\,\bar{B}\,\text{or}\,B\,\text{and}\,\bar{A})=P(A)\cdot P(\bar{B})+P(B)\cdot P(\bar{A})$$
$$=\frac{3}{7}\cdot\frac{2}{7}+\frac{4}{7}\cdot\frac{5}{7}=\frac{26}{49}.$$

One advantage of coming to school in time is one can attend the morning assembly and can hear to the school updates given by the school authority or the principal in the morning assembly.

19. If $x^y=e^{x-y}$, prove that $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$.

Solution

Given that

$$x^y=e^{x-y}.$$

Taking log on both sides, we get

$$\log(x^y)=\log e^{(x-y)}$$
$$\Rightarrow y\log x=(x-y)\log e$$
$$\Rightarrow y\log x=x-y \quad [\because \log e=1]$$
$$\Rightarrow y\log x+y=x$$
$$\Rightarrow y(1+\log x)=x$$
$$\Rightarrow y=\frac{x}{1+\log x}.$$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{(1+\log x)\frac{d}{dx}x-x\frac{d}{dx}(1+\log(x))}{(1+\log x)^2}$$
$$=\frac{(1+\log x)-\cancel{x}\cdot\frac{1}{\cancel{x}}}{(1+\log x)^2}$$
$$=\frac{\cancel{1}+\log x-\cancel{1}}{(1+\log x)^2}$$
$$\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$$

Hence proved.

20. Evaluate:

$$\int \frac{dx}{x(x^3+8)}$$

Solution

Let $I = \int \frac{dx}{x(x^3+8)}$. Therefore,

$$I = \int \frac{dx}{x(x^3+8)} = \int \frac{1}{x^4\left(1+\frac{8}{x^3}\right)} dx.$$

Substituting $1+\frac{8}{x^3} = t,$ and differentiating with respect to x, we get

$$\frac{-24}{x^4} dx = dt$$

$$\Rightarrow \frac{dx}{x^4} = \frac{-dt}{24}$$

Therefore,

$$I = \frac{-1}{24}\int \frac{dt}{t}$$

$$= \frac{-1}{24}\log|t| + c$$

$$= \frac{-1}{24}\log\left|1+\frac{8}{x^3}\right| + c$$

$$= \frac{-1}{24}\log\left|\frac{x^3+8}{x^3}\right| + c.$$

21. Evaluate:

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$$

Solution

Let us consider that

$$I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx \qquad (1)$$

Replacing $x \to \pi - x$, we get

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$$

That is,

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx \qquad (2)$$

Adding Eqs. (1) and (2), we get

$$2I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

$$= -\pi\int_1^{-1} \frac{dt}{1+t^2}.$$

Substituting $\cos x = t$, we get

$$-\sin x dx = dt$$

$$\begin{array}{ll} \text{As } x \to 0 & \text{As } x \to \pi \\ \cos 0 = t & \cos \pi = t \\ \Rightarrow t = 1 & t = -1 \end{array}$$

this implies

$$2I = \pi \int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi[\tan^{-1} t]_{-1}^{1}$$

$$= \pi[\tan^{-1} 1 - \tan^{-1}(-1)]$$

$$I = \frac{\pi}{2}\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$$

$$= \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

22. If $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$, then find the value of λ, so that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are perpendicular vectors.

Solution

Given $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$ then $\vec{p} + \vec{q} = (6\hat{i} + (3+\lambda)\hat{j} - 8\hat{k})$ and $\vec{p} - \vec{q} = (4\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k})$ is $(\vec{p} + \vec{q}) \perp (\vec{p} - \vec{q})$. Therefore,

$$(\vec{p} + \vec{q}) \cdot (\vec{p} - \vec{q}) = 0$$

That is,

$$(6\hat{i} + (3+\lambda)\hat{j} - 8\hat{k}) \cdot (4\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k}) = 0$$

$$\Rightarrow 24 + (\lambda^2 - 9) - 16 = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1.$$

SECTION C

Question numbers 23 to 29 carry 6 marks each.

23. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

Solution

Equation of plane though the intersection of two given planes is $(\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6) + \lambda(\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k})) = 0$, which implies that

$$\vec{r} \cdot \{(1+3\lambda)\hat{i} + (3-\lambda)\hat{j} - 4\lambda\hat{k}\} = 6 \qquad (1)$$

Given that the perpendicular distances of the required plane (1) from origin is unity. That is,

$$\left| \frac{-6}{\sqrt{(1+3\lambda)^2 + (3-\lambda)^2 + (-4\lambda)^2}} \right| = 1$$

$$\Rightarrow 36=(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2$$
$$\Rightarrow 36=1+9\lambda^2+6\lambda+9+\lambda^2-6\lambda+16\lambda^2$$
$$\Rightarrow 36=26\lambda^2+10$$
$$\Rightarrow 26\lambda^2=26$$
$$\Rightarrow \lambda^2=1$$
$$\Rightarrow \lambda=\pm 1.$$

Therefore, the equations of planes are

$$\vec{r}\cdot(4\hat{i}+2\hat{j}-4\hat{k})=6 \text{ and } \vec{r}\cdot(2\hat{i}-4\hat{j}-4\hat{k})+6=0$$

or

$$\vec{r}\cdot(2\hat{i}+\hat{j}-2\hat{k})=3 \text{ and } \vec{r}\cdot(\hat{i}-2\hat{j}-2\hat{k})+3=0.$$

OR

Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}.(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}.(3\hat{i}+\hat{j}+\hat{k})=6$.

Solution

The direction of the line parallel to the planes

$$\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=5 \text{ and } \vec{r}\cdot(3\hat{i}+\hat{j}+\hat{k})=6$$

is

$$\vec{b}=(\hat{i}-\hat{j}+2\hat{k})\times(3\hat{i}+\hat{j}+\hat{k})$$
$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ 3 & 1 & 1\end{vmatrix}$$
$$\Rightarrow \vec{b}=-3\hat{i}+5\hat{j}+4\hat{k}$$

Therefore, the equation of line is

$$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(-3\hat{i}+5\hat{j}+4\hat{k}).$$

24. In a hockey match, both teams A and B scored same number of goals up to the end of the game, so to decide the winner, the referee asked both the captains to throw a die alternately and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of winning the match and state whether the decision of the referee was fair or not.

Solution

The probability of success $(p)=\dfrac{1}{6}$. Hence, the probability of failure is $(q)=\dfrac{5}{6}$. Team A can win in first throw or third throw or fifth throw and so on. Therefore,

$$p(A\text{winning})=p+q\cdot q\cdot p+q\cdot q\cdot q\cdot q\cdot p+\cdots$$

$$= \frac{1}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \times \frac{1}{6} + \cdots$$

$$= \frac{1}{6}\left[\frac{1}{1-\left(\frac{5}{6}\right)^2}\right] \qquad \left(\because S_\infty = \frac{a}{1-r}\right)$$

$$= \frac{1}{6} \times \frac{36}{11}$$

$$= \frac{6}{11}$$

and

$$p(B\,\text{winning}) = 1 - \frac{6}{11} = \frac{5}{11}.$$

Since the probabilities are not the same, the decision of the referee is not fair.

25. A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital, respectively, which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers and 1 unit of capital are required to produce one unit of B. If A and B are priced at Rs.100 and Rs.120 per unit, respectively, how should he use his resources to maximize the total revenue? Form the above as an LPP and solve graphically.

Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate?

Solution

Let us consider that x units of A and y units of B be produced. Therefore, the total revenue is

$$Z = 100x + 120y$$

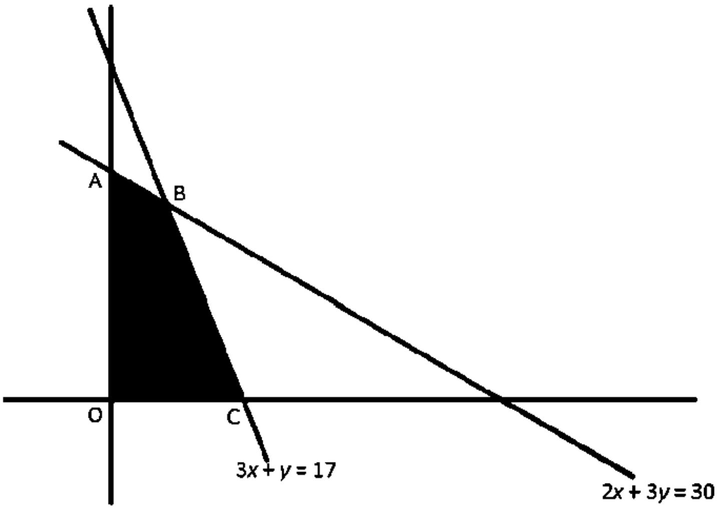
For the production in the given units A and B, the total number of workers used is $2x + 3y$.
The total capital used in the production in units A and B is $3x + y$.
Therefore, according to the given data, we have

$$2x + 3y \le 30$$
$$3x + y \le 17$$
$$x \ge 0, y \ge 0$$

The vertices of feasible region are A(0, 10), B(3, 8), $C\left(\frac{17}{3}, 0\right)$, O(0, 0). Now,

$$Z_O = 0;$$
$$Z_A = 1200;$$
$$Z_B = 1260;$$
$$Z_C = \frac{1700}{3}.$$

Therefore, the maximum total income (see the following figure) Z is at $B(3, 8)$, that is, Rs. 1260.



26. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution

Let point A be (x, y). Therefore, AB = 2y and BC = 2x. The area ABCD (see the following figure) is given by

$$\begin{aligned} A &= \text{AB} \times \text{BC} \\ &= 2x \times 2y \\ &= 4xy \\ A^2 &= 16x^2y^2 \end{aligned}$$

Thus,

$$S = 16x^2y^2 \qquad (1)$$

Therefore,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=\frac{b^2}{a^2}(a^2-x^2)$$

Hence,

$$S = 16x^2\left(\frac{b^2}{a^2}(a^2-x^2)\right)$$

Differentiating with respect to x, we get

$$\frac{dS}{dx}=\frac{16b^2}{a^2}(2a^2x-4x^3)$$

Substituting $\frac{ds}{dx}=0$, we get

$$\frac{16b^2}{a^2}(2a^2x-4x^3)=0 \Rightarrow x=\frac{a}{\sqrt{2}}$$

Therefore,

$$\frac{d^2S}{dx^2}=\frac{16b^2}{a^2}(2a^2-12x^2)$$

Now,

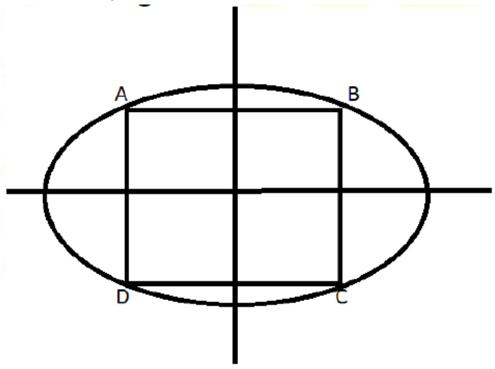
$$\left.\frac{d^2S}{dx^2}\right]_{x=\frac{a}{\sqrt{2}}} = \frac{16b^2}{a^2}(2a^2 - 6a^2)$$

$$= \frac{16b^2}{a^2}\cdot(-4a^2)$$

$$= -64b^2 < 0.$$

Therefore, the area is maximum when

$$x = \frac{a}{\sqrt{2}} \text{ and } y = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{2}} = \frac{b}{2}.$$



OR

Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which pass through the point $\left(\frac{4}{3}, 0\right)$.

Solution

On the curve, let us consider that the tangent be at point (x_1, y_1). Therefore,

$$3x_1^2 - y_1^2 = 8 \qquad (1)$$

The slope of the tangent is

$$\left.\frac{dy}{dx}\right|_{(x_1,\, y_1)}$$

The equation of the curve is given by $3x^2 - y^2 = 8$. Therefore,

$$6x - 2y\left(\frac{dy}{dx}\right) \Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,\, y_1)} = \frac{3x_1}{y_1}$$

The equation of the line passing through point (x_1, y_1) whose slope is $\frac{3x_1}{y_1}$ is given by

$$Y - y_1 = \frac{3x_1}{y_1}(X - x_1)$$

The point $\left(\frac{4}{3}, 0\right)$ is located on

$$Y - y_1 = \frac{3x_1}{y_1}(X - x_1)$$

Therefore,

$$0 - y_1 = \frac{3x_1}{y_1}\left(\frac{4}{3} - x_1\right)$$

$$\Rightarrow y_1^2 - 3x_1^2 + 4x_1 = 0 \qquad (2)$$

On solving Eqs. (1) and (2), we get $4x_1 = 8$ or $x_1 = 2$. Therefore,

$$y_1^2 = \sqrt{3x_1^2 - 8} = \pm 2.$$

Therefore, the equations of the tangents are given by

$$y - 3x + 4 = 0;$$
$$y + 3x - 4 = 0.$$

27. The management committee of a residential colony decided to award some of its members (say, x) for honesty, some (say, y) for helping others and some others (say, z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.

Solution

We have

$$x + y + z = 12;$$
$$3(y + z) + 2x = 33.$$

This implies that

$$2x + 3y + 3z = 33$$
$$x + z = 2y$$
$$x + z = 2y$$
$$\Rightarrow x - 2y + z = 0.$$

The above linear equation can be rewritten as

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}} \underset{X =}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}}$$

$$|A| = 1\begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$$
$$= (3+6) - (2-3) + (-4-3)$$
$$= 9 + 1 - 7$$
$$= 3 \neq 0.$$

Therefore, A is invertible, that is,

$$X = A^{-1}B.$$

Thus,

$$\text{adj } A = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj} A}{|A|} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 108-99 \\ 12 \\ -84+99 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\Rightarrow x = 3, y = 4, z = 5$$

Therefore, the number of awardees for each category, namely, honesty, cooperation and supervision is 3, 4 and 5, respectively.

One more value which the management of the colony must include for awards can be “sincerity towards duty”.

28. Find the area of the region $\{(x, y) : y^2 \leq 6ax \text{ and } x^2 + y^2 \leq 16a^2\}$ using method of integration.

Solution

It is clear that $x^2 + y^2 = 16a^2$ represents the equation of circle having radius $4a$ and centre at (0, 0). Also $y^2 = 6ax$ represents the equation of the parabola having vertex (0, 0). Now, solving

$$x^2 + y^2 = 16a^2 \text{ and } y^2 = 6ax$$

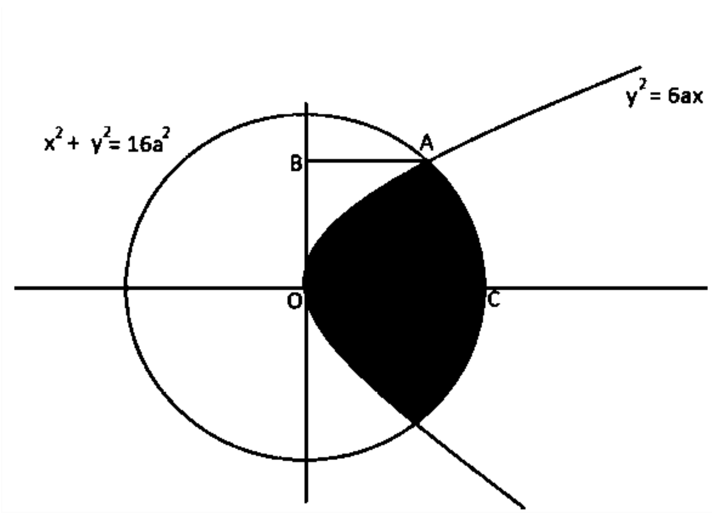
to get the point of intersection of the two curves as $x = 2a$ and $x = –8a$ (rejected), respectively.

The required area of the shaded region (of the figure shown below) is

$$2\left[\int_0^{2a} \sqrt{6a}\sqrt{x}dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2}\,dx\right] = 2\left[2\sqrt{6}\sqrt{a}\,\frac{x^{3/2}}{3}\right]_0^{2a} + 2\left[\frac{x}{2}\sqrt{16a^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$= 2\left(\frac{8\sqrt{3}a^2}{3}\right) + 2\left(4\pi a^2 - 2\sqrt{3}a^2 - \frac{4}{3}\pi a^2\right)$$

$$= \frac{4a^2}{3}[4\pi + \sqrt{3}] \text{ sq.units.}$$



29. Show that the differential equation $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when $x = 1$.

Solution

We have

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0 \qquad (1)$$

can be written as

$$x\,dy = -\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx$$

$$\frac{dy}{dx} = \frac{y - x\sin^2\left(\frac{y}{x}\right)}{x}$$

$$= \frac{y}{x} - \sin^2\left(\frac{y}{x}\right)$$

$$= g\left(\frac{y}{x}\right)$$

Hence, Eq. (1) is a homogeneous differential equation. Substituting $y = vx$, we get

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = v - \sin^2 v$$

Therefore,

$$x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow -\int \frac{dv}{\sin^2 v} = \int \frac{dx}{x}$$
$$\Rightarrow -\int \text{cosec}^2 v\, dv = \log|x| + c$$
$$\Rightarrow -(\cot v) = \log|x| + c$$
$$\Rightarrow \cot v = \log|x| + c$$
$$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|x| + c$$

At $y = \frac{\pi}{4}$ and $x = 1$, we have

$$\cot\left(\frac{\pi}{4}\right) = \log|1| + c$$
$$\Rightarrow c = 1$$

Therefore, the particular solution of the given differential equation is

$$\cot\left(\frac{y}{x}\right) = \log|x| + 1.$$

