CBSE 2014

MATHEMATICS

SECTION A

Question numbers 1 to 10 carry 1 mark each.

1. If $R = \{(x, y) : x + 2y = 8\}$ is a relation of N, write the range of R.

Solution: $R = \{(x, y)! \ x + 2y = 8\}$ a relation of N.

$2y = 8 - x$

$$y = 4 - \frac{x}{2}$$

$\frac{x}{2}$ must be Integer So x Can be, 2, 4, 6

$$x = 2 \Rightarrow y = 4 - 1$$
$$x = 4 \Rightarrow y = 4 - \frac{4}{2} = 2$$
$$x = 6y \Rightarrow y = 4 - \frac{6}{2} = 1$$

Range is {1,2,3}

2. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$, $xy < 1$, then write the value of $x + y + xy$.

Solution:

$$\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$$
$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{4}, \quad (xy < 1)$$
$$\frac{x+y}{1-xy} = 1 \Rightarrow x + y = 1 - xy$$
$$\boxed{x + y + xy = 1}$$

3. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix.

Solution:

A is a square matrix

$A^2 = A$

$7A - (I + A)^3 = 7A - (I + A)(I + A)(I + A)$

$= 7A - (II + IA + AI + A^2)(I + A)$

$= 7A - (I + A + A + A)(I + A)$

$= 7A - (I + 3A)(I + A)$

$= 7A - (II + 3IA + AI + 3AA)$

$= 7A - (I + 4A + 3A)$

$= 7A - (I + 7A) = -I$

4. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of $x + y$.

Solution:

$$\begin{bmatrix} x-y & z \\ 2x-y & \omega \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$$

$$x - y = -1 \quad \& \quad z = 4$$
$$2x - y = 0 \quad \& \quad \omega = 5$$
$$y = 2x$$

$$x - 2x = -1 \Rightarrow x = 1$$
$$y = 2$$

$x + y = 3$

5. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x.

Solution:

$$\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$$
$$\Rightarrow 12x + 14 = 32 - 42$$
$$\Rightarrow 12x = -24$$
$$\Rightarrow x = -2$$

6. If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$.

Solution:

$$f(x) = \int_0^x t \sin t \, \mathrm{d}t$$

$$f(x) = -t \cos t \Big|_0^x + \int_0^x \cos t \, \mathrm{d}t \qquad \text{(By Parts)}$$

$$= -x \cos x + \sin t \Big|_0^x$$

$$f(x) = -x \cos x + \sin x$$

differentiate w.r.t x

$f'(x) = -x(-\sin x) - \cos x + \cos x = x\sin x$

$f'(x) = x\sin x$

7. Evaluate:

$$\int_2^4 \frac{x}{x^2+1}dx$$

Solution:

$$I = \int_2^4 \frac{x}{x^2+1}\mathrm{d}x = \frac{1}{2}\int_2^4 \frac{2x}{x^2+1}\mathrm{d}x$$

$$I = \frac{1}{2}\ln\left(x^2+1\right)\Big|_2^4 = \frac{1}{2}(\ln 17 - \ln 5) = \frac{1}{2}\ln\left(\frac{17}{5}\right)$$

8. Find the value of 'p' for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel.

Solution:

$$\text{let } \bar{a} = 3\hat{i}+2\hat{j}+9\hat{k}$$
$$\bar{b} = \hat{i}-2p\hat{j}+3\hat{k}$$

$\bar{a}$ & $\bar{b}$ are two parallel vector then

$\bar{a} = \lambda\bar{b}$

$(3i + 2j + 9k) = \lambda(i - 2pj + 3k)$

equate Coefficient

$$\lambda = 3, \quad \lambda = -2\lambda P, \quad 9 = 3\lambda$$
$$\lambda P = -1 \qquad \lambda = 3$$
$$P = \frac{-1}{\lambda}$$
$$P = \frac{-1}{3}$$

9. Find $\vec{a}.(\vec{b}\times\vec{c})$, if $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}+2\hat{k}$.

Solution:

$$\bar{a} = 2\hat{i}+\hat{j}+3\hat{k} \quad ,\bar{b} = -\hat{i}+2\hat{j}+\hat{k} \quad ,\bar{c} = 3\hat{i}+\hat{j}+\hat{k}$$

$$\bar{a}\cdot\left(\bar{b}\times\bar{c}\right) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 2(4-1)-1(-2-3)+3(-1-6)$$
$$= 6+5-21 = -10$$

10. If the cartesian equations of a line are $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line.

Solution:

$$\frac{x-3}{-5}=\frac{y+4}{7}=\frac{z-3}{2}$$
$$\frac{x-3}{-5}=\frac{y+4}{7}=\frac{z-3}{2}=\lambda \ \ (\text{Constant})$$

Any point on this line is $(3-5\lambda, -4+7\lambda, 3+2\lambda)$

vector equation of line is

$$\bar{r}=\left(3\hat{i}-4\hat{j}+3\hat{k}\right)+\lambda\left(-5\hat{i}+7\hat{j}+2\hat{k}\right)$$

SECTION B

Question numbers 11 to 22 carry 4 marks each.

11. If the function $f : R \to R$ be given by $f(x) = x^2 + 2$ and $g : R \to R$ be given by $g(x)=\frac{x}{x-1},\ x\neq 1,$ find fog and gof and hence find fog (2) and gof (−3).

Solution:

$$f\,!R\to R\Rightarrow f(x)=x^2+2$$
$$g\,!R\to R\Rightarrow g(x)=\frac{x}{x-1} \qquad x\neq 1$$
$$fog(x)=f(g(x))=g^2(x)+2=\frac{x^2}{(x-1)^2}+2 \qquad (x\neq 1)$$
$$gof(x)=g(f(x))=\frac{f(x)}{f(x)-1}=\frac{x^2+2}{x^2+1}$$
$$fog(2)=\frac{4}{1}+2=6$$
$$gof(-3)=\frac{3^2+2}{3^2+1}=\frac{11}{10}$$

12. Prove that

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ \ \frac{-1}{\sqrt{2}}\leq x\leq 1$$

Solution:

let $y=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

let $x=\cos\theta$, $\theta\in[0,\pi]$

$$y=\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta/2-\sin\theta/2}{\cos\theta/2+\sin\theta/2}\right)$$

$$\left(\begin{array}{l}\text{use}\\ \cos\theta=1-2\sin^2\theta/2\\ \qquad=2\cos^2\theta/2-1\\ \theta/2\in[0,\pi/2]\end{array}\right)$$

$$=\tan^{-1}\left(\tan\left(\pi/4-\theta/2\right)\right)$$

$$y=\frac{\pi}{4}-\frac{\theta}{2}$$

$$y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

OR

If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, find the value of x.

Solution:

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\left(\frac{x-2}{x-4}\right)\left(\frac{x+2}{x+4}\right)}\right)=\frac{\pi}{4}$$

$$\left(\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\left(\frac{x-2}{x-4}\right)\left(\frac{x+2}{x+4}\right)}\right)=\tan\pi/4$$

$$\frac{\left(x^2+2x-8\right)+\left(x^2-2x-8\right)}{-12}=1$$

$$2x^2-16=-12\Rightarrow x^2-8=-6$$

$$\Rightarrow x^2=2$$

$$\Rightarrow x=\pm\sqrt{2}$$

13. Using properties of determinants, prove that

$$\begin{vmatrix}x+y & x & x\\ 5x+4y & 4x & 2x\\ 10x+8y & 8x & 3x\end{vmatrix}=x^3$$

Solution:

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = \begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix} + \begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix}$$

$$= x^3 \begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix} + x^2 y \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix}$$

$$= x^3 \begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix} + 0$$

$$= x^3 \begin{vmatrix} 1 & 0 & 0 \\ 5 & -1 & -3 \\ 10 & -2 & -7 \end{vmatrix} \quad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$$= x^3(+7-6) = x^3$$

14. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^{\theta}$ (sin θ – cos θ) and $y = ae^{\theta}$ (sin θ + cos θ).

Solution:

$$x = ae^{\theta}(\sin\theta - \cos\theta) \quad , \quad y = ae^{\theta}(\sin\theta + \cos\theta)$$

$$\frac{dx}{d\theta} = ae^{\theta}(\sin\theta - \cos\theta) + ae^{\theta}(\cos\theta + \sin\theta) = 2ae^{\theta}\sin\theta$$

$$\frac{dy}{d\theta} = ae^{\theta}(\sin\theta + \cos\theta) + ae^{\theta}(\cos\theta - \sin\theta) = 2ae^{\theta}\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta} = \cot\theta$$

$$\left.\frac{dy}{dx}\right)_{\theta=\pi/4} = 1$$

15. If $y = Pe^{ax} + Qe^{bx}$, show that

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$$

Solution:

$$y = Pe^{ax} + Qe^{bx}$$

$$\frac{dy}{dx} = aPe^{ax} + bQe^{bx} \qquad (\text{differentiate w.r.t to } x)$$

$$\frac{d^2y}{dx^2} = a^2Pe^{ax} + b^2Qe^{bx}$$

$$\frac{d^2y}{dx^2} - (a+b)\frac{dx}{dy} + aby = \left(a^2Pe^{ax} + b^2Qe^{bx}\right) - (a+b)\left(aPe^{ax} + bQe^{bx}\right) + ab\left(Pe^{ax} + Qe^{bx}\right)$$

$$= a^2Pe^{ax} + b^2Qe^{bx} - a^2Pe^{ax} - abQe^{bx} - baPe^{ax} - b^2Qe^{bx} + abPe^{ax} + Qabe^{bx} = 0$$

16. Find the value(s) of x for which $y = [x(x-2)]^2$ is an increasing function.

Solution:

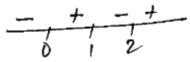
$y = x^2(x-2)^2$

differentiate w.r.t x

$$\frac{dy}{dx} = 2x(x-2)^2 + 2x^2(x-2) = 2x(x-2)(x-2+x)$$

$$= 4x(x-1)(x-2)$$

Sign schemes of $\frac{dy}{dx}$



$f'(x) \geq 0$

y is increasing $[0,1] \cup (2,\omega)$

OR

Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$.

Solution:

Equation of Hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Equation of tangent to Hyperbola at point $P\left(\sqrt{2}a, b\right)$

$$\frac{x \cdot \sqrt{2}a}{a^2} - \frac{y \cdot b}{b^2} = 1 \Rightarrow \sqrt{2}\frac{x}{a} - \frac{y}{b} = 1$$

Slope of tangent $= \sqrt{2}\frac{b}{a}$

Slope of normal $= -\frac{a}{b\sqrt{2}}$

Equation of normal at point $P\left(\sqrt{2}a, b\right)$

$$y - b = -\frac{a}{b\sqrt{2}}\left(x - \sqrt{2}a\right)$$
$$b\sqrt{2}y - b^2\sqrt{2} = -ax + \sqrt{2}a^2$$
$$ax + \sqrt{2}by = \sqrt{2}\left(a^2 + b^2\right)$$

17. Evaluate:

$$\int_0^{\pi} \frac{4x \sin x}{1 + \cos^2 x} dx$$

Solution:

Let

$$I=\int_0^{\pi}\frac{4x\sin x}{1+\cos^2 x}dx$$

$$I=\int_0^{\pi}\frac{4(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx \qquad (u\sin g\ porperty)$$

$$2I=\int_0^{\pi}\frac{4\pi\sin x}{1+\cos^2 x}dx$$

$$I=2\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx=-2\int_0^{\pi}\frac{-(-\sin x)}{1+\cos^2 x}dx$$

$$=-2\pi\tan^{-1}(\cos x)\Big|_0^{\pi}$$

$$=-\pi^2\left(\tan^{-1}(\cos\pi)-\tan^{-1}(\cos 0)\right)$$

$$=-2\pi\left(-\pi/2\right)=\pi^2$$

OR

Evaluate:

$$\int\frac{x+2}{\sqrt{x^2+5x+6}}dx$$

Solution:

$$I=\int\frac{(x+2)}{\sqrt{x^2+5x+6}}dx$$

$$=\frac{1}{2}\int\frac{(2x+5)}{\sqrt{x^2+5x+6}}dx-\frac{1}{2}\int\frac{dx}{\sqrt{x^2+5x+\frac{25}{4}+6-\frac{25}{4}}}$$

$$=\frac{1}{2}\int\frac{(2x+5)dx}{\sqrt{x^2+5x+6}}-\frac{1}{2}\int\frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

$$=\frac{1}{2}\frac{\left(x^2+5x+6\right)^{-1/2+1}}{-\frac{1}{2}+1}-\frac{1}{2}\ln\left|\left(x+\frac{5}{2}\right)+\sqrt{\left(x+\frac{5}{2}\right)^2+\frac{1}{4}}\right|+C$$

$$=\sqrt{x^2+5x+6}-\frac{1}{2}\ln\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+C$$

18. Find the particular solution of the differential equation $\frac{dy}{dx}=1+x+y+xy,$ given that y = 0 when x = 1.

Solution:

$$\frac{dy}{dx}=1+x+y+xy$$
$$\frac{dy}{dx}=(1+x)+y(1+x)$$
$$\frac{dy}{dx}=(1+x)(1+y)$$
$$\int\frac{dy}{y+1}=\int(x+1)dx+c \quad \text{(Variable Separable method)}$$
$$\ln(y+1)=\left(\frac{x^2}{2}+x\right)+c$$
at $x=1, y=0$
$$\ln 1=\left(\frac{1}{2}+1\right)+c \Rightarrow c=-3/2$$
$$\ln(y+1)=\left(\frac{x^2+2x}{2}\right)-\frac{3}{2}$$
$$\ln(y+1)=\frac{x^2+2x-3}{2}$$
Solution is $y=e^{\left(\frac{x^2+2x-3}{2}\right)}-1$

19. Solve the differential equation $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$.

Solution:

$$\left(1+x^2\right)\frac{dy}{dx}+y=e^{\tan^{-1}x}$$
$$\frac{dy}{dx}+\left(\frac{1}{1+x^2}\right)y=\frac{e^{\tan^{-1}x}}{\left(1+x^2\right)}$$
linear differential equation.

Integrating factor = $e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$
$$\int d\left(ye^{\tan^{-1}x}\right)=\int\frac{e^{\tan^{-1}x}\cdot e^{\tan^{-1}x}}{1+x^2}dx+C$$
$$ye^{\tan^{-1}x}=\int\frac{e^{\tan^{-1}x}\cdot e^{\tan^{-1}x}}{1+x^2}dx+C$$
$$e^{\tan^{-1}x}=t$$
$$e^{\tan^{-1}x}\cdot\frac{1}{1+x^2}dx=dt$$
$$\int\frac{e^{\tan^{-1}x}\cdot e^{\tan^{-1}x}}{1+x^2}dx=\int t\,dt=\frac{t^2}{2}+C$$
$$ye^{\tan^{-1}x}=\frac{\left(e^{\tan^{-1}x}\right)^2}{2}+C$$
$$y=\frac{e^{\tan^{-1}x}}{2}+C\,e^{-\tan^{-1}x}$$

20. Show that the four points *A*, *B*, *C* and *D* with position vectors $4\hat{i}+5\hat{j}+\hat{k},\ -\hat{j}-\hat{k},\ 3\hat{i}+9\hat{j}+4\hat{k}$ and $4(-\hat{i}+\hat{j}+\hat{k})$ respectively are coplanar.

Solution:

Position vector of 4 Points are

$$\overline{OA}=4i+5j+k,\ \overline{OB}=-j-k,\ \overline{OC}=3i+9j+4k,\ \overline{OD}=-4i+4j+4k$$

$$\overline{AB}=\overline{OB}-\overline{OA}=-j-k-4i-5j-k=-4i-6j-2k$$
$$\overline{AC}=\overline{OC}-\overline{OA}=3i+9j+4k-4i-5j-k=-i+4j+3k$$
$$\overline{AD}=\overline{OD}-\overline{OA}=-4i+4j+4k-4i-5j-k=-8i-j+3k$$

$\overline{AD}$, $\overline{AC}$ & $\overline{AD}$ vector are coplanar than scalar triple product these vector is 0.

$$\left[\overline{AB}\quad \overline{AC}\quad \overline{AD}\right]=\overline{AB}\cdot\left(\overline{AC}\times\overline{AD}\right)=0$$
$$\begin{vmatrix}-4 & -6 & -2\\ -1 & 4 & 3\\ -8 & -1 & 3\end{vmatrix}=-4(15)+6(21)-2(33)$$
$$=-132+132=0$$

OR

The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$.

Solution:

$$\overline{a}=i+j+k$$
$$u+\overline{d}=\overline{b}+\overline{c}=2i+4j-5k+\lambda i+2j+3k$$
$$=(2+\lambda)i+6j-2k$$

Unit vector $\hat{d}=\dfrac{(\partial+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(\partial+\lambda)^2+36+4}}$

$$\overline{a}\cdot\hat{d}=1\Rightarrow\frac{(i+j+k)\cdot((\partial+\lambda)i+6j-2k)}{\sqrt{(\partial+\lambda)^2+40}}=1$$
$$(\partial+\lambda)+6-2=\sqrt{(\partial+\lambda)^2+40}$$
$$((\partial+\lambda)+4)^2=(\partial+\lambda)^2+40$$
$$(\partial+\lambda)^2+16+8(\partial+\lambda)^2=(\partial+\lambda)^2+40$$
$$32+8\lambda=40\Rightarrow 8\lambda=8$$
$$\Rightarrow\lambda=1$$
$$\hat{d}=\frac{3i+6j-2k}{\sqrt{9+40}}=\left(\frac{3i+6j-2k}{7}\right)$$

21. A line passes through (2, −1, 3) and is perpendicular to the lines

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and

$\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$. Obtain its equation in vector and cartesian form.

Solution:

Line passing through (2,−1,3) is

$$\bar{r} = (2i - j + 3k) + \lambda\left(l\hat{i} + mj + n\hat{k}\right)$$

l, *m*, n are direction of line which is perpendicular to given lines

$$\bar{r} = (i + j - k) + \lambda(2i - 2j + k) \;\;\&\;\; \bar{r} = (2i + j - 3k) + \mu(i + 2j + 2k)$$

Dot product of direction is zero.

$$2l - 2m + n = 0$$
$$l + 2m + 2n = 0$$
$$3l + 3n = 0 \Rightarrow l = -n$$
$$2l - 2m - l = 0 \Rightarrow l = 2m$$

Direction ratio of required line is

$$\left(l, \tfrac{l}{2}, -l\right) = \left(1, \tfrac{1}{2}, -1\right)$$
$$= (2, 1, -2)$$

vector equation of line is

$$\bar{r} = (2i - j + 3k) + \lambda(2i + j - 2k)$$

Cartesian equation of line is

$$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-2}{2}$$

22. An experiment succeeds thrice as often as it fails. Find the probability that in the next five trials, there will be at least 3 successes.

Solution:

Let the chances of success of an experiment is = p

& the chances of an experiment is = q

$p + q = 1$

$p = 3q$

$$4q = 1 \Rightarrow q = \frac{1}{4}$$
$$p = \tfrac{3}{4}$$

Probability of at least 3 success out of 5 trial is

$$= {}^5C_3 p^3 q^2 + {}^5C_4 p^4 q + {}^5C_5 p^5$$
$$= {}^5C_3 \frac{27}{45} + {}^5C_4 \frac{81}{45} + {}^5C_5 \frac{5}{45}$$
$$= \frac{459}{572}$$

SECTION C

Question numbers 23 to 29 carry 6 marks each.

23. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ` $x$ each, ` y each and ` $z$ each for the three respective values to 3, 2 and 1 students respectively with a total award money of ` 1,600. School B wants to spend ` 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is < 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award.

Solution:

Name of school of their no of Student	Award for Sincenty (x)	Award for Truthful ness (y)	Award for Helpless (z)
A	3	2	1
B	4	1	3
1 Student for each value	1	1	1

School A award the money $\Rightarrow$ $3x + 2y + z = 1600$ (1)

School B award the money $\Rightarrow$ $4x + y + 3z = 2300$ (2)

Total amount of award for one Prize $\Rightarrow$ $x + y + z = 900$ (3)

On each value is

$$\text{let } P = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad , \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$Q = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$$

$$PX = Q$$

$$\text{Det}(P) = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3(1-3) - 2(4-3) + 1(4-1)$$

$$= -6 - 2 + 3$$
$$= -5 \neq 0$$

Matrix P is non Singular matrix

$X = P^t Q$

Cofactor is of P is

$$C_{11} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1-3 = -2, C_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1, C_{31} = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = +5$$

$$C_{12} = -\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = -(4-3) = -1, C_{22} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2, C_{32} = \begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} = -5$$

$$C_{13} = \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} = 3 \quad , C_{23} = -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -1, C_{33} = \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = -5$$

$$\text{adj}\,P = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$P^t = \frac{(\text{adj}\,P)}{(\text{Det}\,P)} = -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

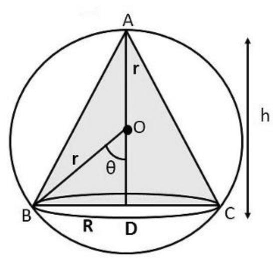
$$P^t = \begin{bmatrix} 2/5 & 1/5 & -1 \\ 1/5 & -2/5 & 1 \\ -3/5 & 1/5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 & -1 \\ 1/5 & -2/5 & 1 \\ -3/5 & 1/5 & 1 \end{bmatrix} \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$

$x = 200, y = 300, z = 400$

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

Solution:



ΔOBD

$$\cos\theta = \frac{OD}{OB} = \frac{OD}{r} \quad , \quad \sin\theta = \frac{BD}{OB}$$

$$OD = r\cos\theta \qquad BD = r\sin\theta$$

Volume of right Circular Cone is

$V = \frac{1}{3}\pi R^2 h$

$R \rightarrow$ Radius of Cone

$h \rightarrow$ height of Cone

$h \rightarrow r + r\cos\theta$

$R = r\sin\theta$

$V = \frac{1}{3}\pi r^2 \sin^2\theta(r + r\cos\theta)$

$V = \frac{1}{3}\pi r^3(\sin^2\theta + \sin^2\theta\cos\theta)$

$\frac{\mathrm{dv}}{\mathrm{d}\theta} = \frac{1}{3}\pi r^3(2\sin\theta\cos\theta + 2\sin\theta\cos^2\theta - \sin^3\theta)$ (differentiatic w.r.t θ)

$\frac{\mathrm{dv}}{\mathrm{d}\theta} = \frac{1}{3}\pi r^3 \sin\theta(2\cos\theta + 2\cos^2\theta - \sin^2\theta)$

$\frac{\mathrm{dv}}{\mathrm{d}\theta} = 0 \Rightarrow 2\cos\theta + 2\cos^2\theta - 1 + \cos^2\theta = 0$

$3\cos^2\theta + 2\cos\theta - 1 = 0$

$(3\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta + 170 \Rightarrow \cos\theta = 1/3$

$\sin\theta = \frac{2/2}{3}$

hieght $= r + \frac{r}{3} = \frac{4r}{3}$

Volume of Cone $= \frac{1}{3}\pi R^2 h$

$= \frac{1}{3}\pi(r\sin\theta)\ \frac{4r}{3}$

$= \frac{4}{9}\pi r^3 \sin^2\theta$

$= \frac{4}{9}\pi r^3\left(\frac{8}{9}\right)$

$= \frac{32}{81}\pi r^3$

$= \frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{8}{27}$ volume of sphase

25. Evaluate:

$$\int \frac{1}{\cos^4 x + \sin^4 x}\, dx$$

Solution:

Let $\mathrm{I}=\int \frac{1}{\cos^4 x+\sin^4 x}\mathrm{d}x$

$\mathrm{I}=\int \frac{\sec^4 x}{1+\tan^4 x}\mathrm{d}x$

Put $\tan x = t \Rightarrow \sec^2 x\,\mathrm{d}x \cdot \mathrm{d}t$

$$I=\int \frac{\sec^2 x}{1+t^4}\mathrm{d}t=\int \frac{1+t^2}{1+t^4}\mathrm{d}t$$

$$I=\int \frac{(1+1/t^2)}{\left(t^2+\frac{1}{t^2}-2\right)+2}\mathrm{d}t$$

$$I=\int \frac{(1+1/t^2)}{(t-1/t)^2+(\sqrt{2})^2}$$

$$t\frac{1}{t}=z \Rightarrow \left(1+\frac{1}{t^2}\right)\mathrm{d}t=\mathrm{d}z$$

$$I=\int \frac{\mathrm{d}z}{z^2+(\sqrt{2})^2}=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-1/t}{\sqrt{2}}\right)+C$$

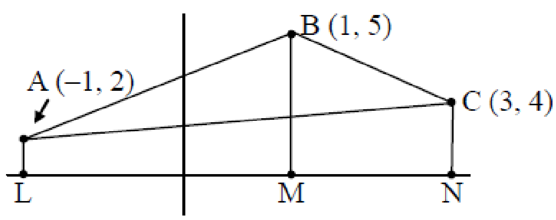
$$I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-\cot x}{\sqrt{2}}\right)+C$$

26. Using integration, find the area of the region bounded by the triangle whose vertices are (−1, 2), (1, 5) and (3, 4).

Solution:

Coordinate of Vertices of ΔABC is

A(−1, 2) B(1, 5) & C(3, 4)



Equation of line AB is

$$y-z=\frac{5-2}{1+1}(x+1)$$

$$2y-4=3x+3 \Rightarrow 3x-2y+7=0$$

$$y=\frac{3x+7}{2}$$

Equation of line AC is

$$y - z = \frac{4-2}{3+1}(x+1)$$
$$4y - 8 = 2x + 2 \Rightarrow 4y = 2x + 10$$
$$y = \frac{x+5}{2}$$

Equation of line BC is

$$y - 4 = \frac{5-4}{1-3}(x-3) \Rightarrow -2y + 8 = x - 3$$
$$\frac{-x+11}{2} = y$$

$$\text{Area of } \Delta\text{ABC is} = \left| \int_{t} \left(\frac{3x+7}{2}\right) dx \int_{-1}^{3} \left(\frac{x+5}{2}\right) dx + \int_{t_1}^{3} \left(\frac{11-x}{2}\right) dx \right|$$

$= 6$ sq. unit

$$\text{Area of } \Delta\text{ABC} = \frac{1}{2}\left| \int_{-1}^{1} (3x+7)dx + \int_{1}^{3} (11-x)dx - \int_{-1}^{3} (x+5)dx \right|$$
$$= \frac{1}{2}\left| \left(\frac{3x^2}{2} + 7x)\right)\Big|_{-1}^{1} + \left(11x - \frac{x^2}{2}\right)\Big|_{1}^{3} + \left(\frac{x^2}{2} + 5x\right)\Big|_{1}^{3} \right|$$
$$= \frac{1}{2}\left| \left(\frac{3}{2} + 7 - \frac{3}{2} + 7\right) \right| + \left(33 - \frac{9}{2} - 11 + \frac{1}{2}\right)$$

$$-\left(\frac{9}{2} + 15 - \frac{1}{2} + 5\right)$$
$$= \frac{1}{2}|14 + 22 - 4 - 4 - 20|$$
$$= \frac{1}{2}|36 - 28| = 4 \quad \text{sq.unit.}$$

27. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Also find the distance of the plane obtained above, from the origin.

Solution: Equation of plane passing through the line of Intersection of two given plane is given by

$$(x + y + z +) + \lambda(2x + 3y + 4z - 5) = 0$$
$$x(1 + 2\lambda) + y(1 + 3\lambda) + z(1 + 4\lambda) - (1 + 5\lambda) = 0$$

This plane is perpendicular to $x - y + z = 0$

$$(1 + 2\lambda) - 1 - 3\lambda + 14\lambda = 0$$
$$1 + 3\lambda = 0 \Rightarrow \lambda = -1/3$$

Hence plane is $\Rightarrow 3x + 3y + 3z - 3 - zx - 3y - 4z + 5 = 0$

$$\boxed{x - z + 2 = 0}$$

Distance of Plane from origin $= \left|\frac{2}{\sqrt{2}}\right| = \sqrt{2}$

OR

Find the distance of the point (2, 12, 5) from the point of intersection of the line $\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$.

Solution:

Eq^n of plane is $\overline{r} \cdot (i - 2j + k) = 0$

$$(xi + yj + zk) \cdot (i - 2j + k) = 0$$

$$x - 2y + z = 0$$

& Eq^n of line is

$\overline{r} = 2i - 4j + 2k + \lambda(si + 4j + 2k)$

$\overline{r} = i(2 + 3\lambda) + j(-4 + 4\lambda) + k(2 + 2\lambda)$

any point on this line is

$(2 + 3\lambda_1 - 4 + 4\lambda_1 2 + 2x)$ lies in the Plane

$2 + 2\lambda + 8 - 8\lambda + 2 + 2\lambda = 0$

$12 - 3\lambda = 0 \Rightarrow \lambda = 4$

Point of Intersection of plane & line is

(11, 12, 10) (2,12,5)

Distance b/w given point ↑ & this point is

$\sqrt{(14-2)^2 + (12-12)^2 + (10-5)^2}$

$= \sqrt{141 + 25}$

$= \sqrt{169} = 13$

28. A manufacturing company makes two types of teaching aids A and B of Mathematics for class XII. Each type of A requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ` 80 on each piece of type A and ` 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Make it as an LPP and solve graphically. What is the maximum profit per week?

Solution: Let no. of pieces of type A manufactured per week =*x*

Let no. of pieces of type B manufactured per week =y

Type of pieces	hour for fabrication	hour for finishing
A	9	1
B	12	3

for A

Maximum no. of fabrication 9 hours ↑ is 180.

$9x+12y \le 180 \Rightarrow \quad 3x+4y \le 60$

Max[m] no. of finishing hour for B is 30.

$x+3y \le 30$

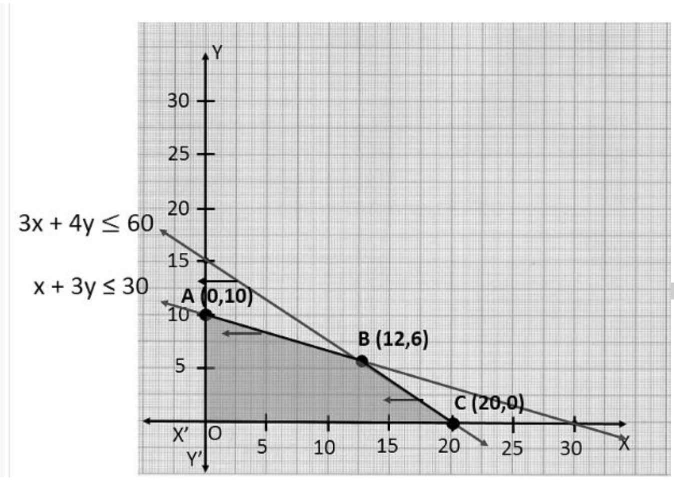
$\underset{(P)}{\text{Profit}} = \ 80x+120y$

Where $x_1 y \ \ge \ 0$

$3x+4y \le 60 \Rightarrow \frac{x}{20}+\frac{y}{15} \le 1$

$x+3y \le 30 \Rightarrow \frac{x}{30}+\frac{y}{10} \le 1$

Graphically Plotting of this



$x+3y \le 30 \Rightarrow 3x+gy \le 90$

$\underline{\quad 3x+4y \le 60}$

$\underline{5y \le 30 \Rightarrow y \le 6}$

$x+18 \le 30 \quad \Rightarrow \quad x \le 12$

$P = 80x+120y$

$Q(12,b) \Rightarrow \text{Pr ofit} + P = 80\times 12+120\times 6 = 960+720 = 1680$

$S(0,10) \Rightarrow P = 80\times 0+120\times 10 = 1200$

$R(20,0) \Rightarrow P = 80\times 20+120\times 0 = 1600$

$T(0,15) \Rightarrow P = 80\times 0+120\times 15 = 1800$

Maximum Profit occur at T(0, 15)

Where Teaching aid from A=0

Teaching aid from B=15

29. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin?

Solution: Number of Coin=3

For coin P $\Rightarrow$ Probability of P Heads is P(H/P) =1

For coin Q $\Rightarrow$ Probability of P Heads is $P(H/Q) = \frac{75}{100} - 3/4$

For coin R $\Rightarrow$ Probability of P Heads is in $P(H/R) = \frac{60}{100} - \frac{3}{5}$

Prob of choosing one Coin out of 3 coins is =1/3

$$P(P) = P(Q) = P(R) = 1/3$$

P(P/H)$\Rightarrow$Prob. Of Head from coin P, same other

Prob of Head P(H)=P(P)·P(H/P)+P(Q)P(H/Q)+P(R) P(H/R)

$$= \frac{1}{3}\left(1 + \frac{3}{4} + \frac{3}{5}\right) = \frac{1}{3}\left(\frac{7}{4} + \frac{3}{5}\right)$$

$$= \frac{1}{3} \cdot \frac{47}{20} = \frac{47}{60}$$

$$P(P/H) = \frac{P(P) \times P(H/P)}{P(H)} = \frac{\frac{1}{3} \cdot 1}{\frac{1}{3} \cdot \frac{47}{20}} = \frac{20}{47}$$

OR

Two numbers are selected at random (without replacement) from the first six positive integers. Let *X* denote the larger of the two numbers obtained. Find the probability distribution of the random variable *X*, and hence find the mean of the distribution.

Solution: First six positive Integer is {1,2,3,4,5,6}

X is greatest no out of chosen of 2no.

X =	2	3	4	5	6

If x=2 $\Rightarrow$ favourable Case is (1,2)

$$P(x = 2) = \frac{1}{{}^6C_2} = \frac{1}{15}$$

If $x = 3$ $\Rightarrow$ favourable Case is

(1,3) & (2,3)

$$P(x=3)=\frac{2}{{}^{6}C_{2}}=\frac{2}{15}$$

If x = 4 $\Rightarrow$ favourable case is (1,4), (2,4), (3,4)

$$P(x=4)=\frac{31}{{}^{6}C_{2}}=\frac{3}{15}$$

If x = s $\Rightarrow$ fav. Case is (1,5), (2,5), (3,5), (4,5)

$$P(x=5)=\frac{4}{{}^{6}C_{2}}=\frac{4}{15}$$

If x = 6 $\Rightarrow$ fav Case is (1,6),(2,6,),(3,6),(4,6),(5,6)

$$P(x=6)=\frac{5}{{}^{6}C_{2}}=\frac{5}{15}$$

Variable X	2	3	4	5	6
Corresponding Probability	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

$$\text{Mean of distribution}=2\cdot\frac{1}{15}+3\cdot\frac{2}{15}+4\cdot\frac{3}{15}+5\cdot\frac{4}{15}+6\cdot\frac{5}{15}$$

$$=\frac{(2+6+12+20+30)}{15}=\frac{70}{15}=\frac{14}{3}$$

