CBSE CLASS 12 EXAMINATION - 2017

Mathematics – Solutions

SECTION-A

1. If for any 2 × 2 square matrix A, $A(\text{adj}A)=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}$, then write the value of $|A|$.

Solution

It is given that $A(\text{adj}A)=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}$

From one of the properties of matrices, we have

$$A(\text{adj}\,A)=|A|I,$$

where I is an identity matrix, which is given by

$$I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Therefore,

$$|A|I=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}$$

$$\Rightarrow |A|I=8\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=8I$$

$$\Rightarrow |A|=8$$

Hence, the value of $|A|$ is 8.

2. Determine the value of k for which the following function is continuous at $x = 3$:

$$f(x)=\begin{cases}\left[\dfrac{(x+3)^2-36}{x-3}\right] & , \quad x\neq 3\\ k & , \quad x=3\end{cases}$$

Solution

- **Case 1:** When $x \neq 3$:

$$f(x)=\frac{(x+3)^2-36}{x-3}$$

Considering the limit form of $f(x)$, we get

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{(x+3)^2-36}{x-3}=\frac{0}{0}$$

Now, factorising the numerator of $f(x)$, we get

$$f(x) = \frac{(x+3)^2 - 36}{x-3} = \frac{x^2 + 9 + 6x - 36}{x-3}$$

$$\Rightarrow f(x) = \frac{x^2 + 6x - 27}{x-3} = \frac{x^2 + 9x - 3x - 27}{x-3}$$

$$\Rightarrow f(x) = \frac{x(x+9) - 3(x+9)}{x-3}$$

$$\Rightarrow f(x) = \frac{(x-3)(x+9)}{x-3} = x + 9$$

Now, taking the limit of $\lim_{x \to 3} f(x)$, we get

$$\lim_{x \to 3} f(x) = x + 9 = 3 + 9$$

That is, $$\lim_{x \to 3} f(x) = 12 \tag{1}$$

- **Case 2:** When $x = 3$:

$$f(x) = k$$

Now, taking the limit form of $f(x)$, we get

$$\lim_{x \to 3} f(x) = k \tag{2}$$

Equating Eqs. (1) and (2), we get $k = 12$.

Therefore, for the value of $k = 12$, the given function is continuous at $x = 3$.

3. Find: $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$

Solution

We have

$$I = \int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx = -\int \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} dx$$

Dividing and multiplying denominator by the factor 2, we get

$$I = -\int \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} dx$$

Using trigonometric identities, we get

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin 2x = 2 \sin x \cos x$$

Therefore,

$$I = -\int \frac{\cos 2x}{\sin(2x/2)} dx = -2\int \frac{\cos 2x}{\sin 2x} dx = -2\int \cot 2x \, dx$$

Using standard integral value of cot(x), we get

$$\int \cot \ ax \, dx = \frac{1}{a} \ln|\sin \ ax| + C$$

Therefore, $I = -2\left[\frac{1}{2} \ln \ |\sin 2x| + C\right] = -\ln \ |\sin 2x| + C$

Hence, the value of the given integral is

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx = -\ln \ |\sin 2x| + C$$

4. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.

Solution

Let us first consider the plane: $2x - y + 2z = 5$

To find a point above the plane, let $x = 0$ and $z = 0 \Rightarrow y = -5$

Therefore, we find that the point $(0, -5, 0)$ lies on plane $2x - y + 2z = 5$

Now, we know that the distance d of a point (x_1, y_1, z_1) from a plane $ax + by + cz + d = 0$ is given by

$$d = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

So, the distance of point $(0, -5, 0)$ from plane $5x - 2.5y + 5z = 20$ is

$$d = \left|\frac{5(0) - 2.5(-5) + 5(0) - 20}{\sqrt{5^2 + (-2.5)^2 + 5^2}}\right|$$

$$\Rightarrow d = \left|\frac{12.5 - 20}{\sqrt{25 + 6.25 + 25}}\right| = \left|\frac{-7.5}{\sqrt{56.25}}\right| = \left|\frac{-7.5}{7.5}\right|$$

$$\Rightarrow d = 1$$

Therefore, the distance between the given two planes is 1.

SECTION-B

5. If A is a skew-symmetric matrix of order 3, then prove that det $A = 0$.

Solution

It is given that A is a skew-symmetric matrix; therefore, we have

$$A^{\mathrm{T}} = -A \qquad (1)$$

From one of the properties of determinants, we have

$$\det(A^{\mathrm{T}}) = \det(A)$$

Thus, from Eq. (1), we get

$$\det(-A) = \det(A) \quad (2)$$

From another property of determinants, if matrix B is of order n and b is a constant, then,

$$\det(bB) = b^n \det(B)$$

Here, A is the matrix of order 3; therefore, from Eq. (2), we get

$$(-1)^3 \det(A) = \det(A)$$

Therefore, $$-1 \det(A) = \det(A)$$

$$\Rightarrow 2 \det(A) = 0$$

$$\Rightarrow \det(A) = 0$$

Hence, it is proved.

6. Find the value of c in Rolle's Theorem for the function $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$.

Solution

Rolle's Theorem states that if $f(x)$ is a continuous function on $[a, b]$ (i.e. closed interval) and differentiable on (a, b) (i.e. open interval) and if $f(a) = f(b)$, then there exists a point c in (a, b) such that $f'(c) = 0$.

For the given function,

$$f(x) = x^3 - 3x$$

we have $a = -\sqrt{3}$ and $b = 0$.

Now, $f(a) = (-\sqrt{3})^3 - 3(-\sqrt{3}) = -3\sqrt{3} + 3\sqrt{3} = 0$

$$f(b) = 0 - 3 \times 0 = 0$$

That is, $$f(a) = f(b)$$

Thus, there exists a point c such that $f'(c) = 0$. Taking derivative of $f(x)$, we have

$$f'(x) = \frac{d}{dx}(x^3 - 3x)$$

$$f'(x) = 3x^2 - 3$$

$$f'(x) = 0 \Rightarrow 3x^2 - 3 = 0$$

Therefore,

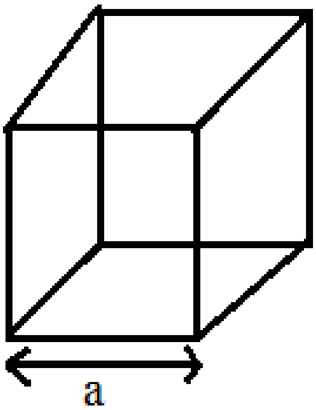
$$3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm\sqrt{1}$$

Therefore, the required value of c is ± 1.

7. The volume of a cube is increasing at a rate of $9\ \text{cm}^3/\text{s}$. How fast is its surface area increasing when the length of an edge is 10 cm?

Solution

Let us consider a cube of edge length a as shown in the following figure:



Volume of cube, $V = a^3$

Surface area of cube, $S = 6a^2$

It is given that $a = 10$ cm and $dV/dt = 9\ \text{cm}^3/\text{s}$

Differentiating S w.r.t. t, we get

$$\frac{dS}{dt} = \frac{d}{dt}(6a^2)$$

$$\frac{dS}{dt} = 2 \times 6a\frac{da}{dt} \qquad (1)$$

Differentiating V w.r.t. t, we get

$$\frac{dV}{dt} = \frac{d}{dt}(a^3)$$

$$\frac{dV}{dt} = 3a^2\frac{da}{dt}$$

Rearranging, it we get

$$\frac{da}{dt} = \frac{1}{3a^2}\frac{dV}{dt}$$

Substituting $\frac{dV}{dt} = 9\ \text{cm}^3/\text{s}$ in this equation, we get

$$\frac{da}{dt} = \frac{1}{3a^2} \times 9 = \frac{3}{a^2}\ \text{cm}^3/\text{s} \qquad (2)$$

Substituting Eq. (2) in Eq. (1), we get

$$\frac{dS}{dt} = 2 \times 6a \times \frac{3}{a^2} = \frac{36}{a}\ \text{cm}^3/\text{s}$$

Substituting $a = 10$ cm in this equation, we get

$$\frac{dS}{dt} = \frac{36\ \text{cm}^3/\text{s}}{10\ \text{cm}} = 3.6\ \text{cm}^2/\text{s}$$

Therefore, the surface area is increasing at the rate of $3.6\ \text{cm}^2/\text{s}$.

8. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on $\mathbb{R}$.

Solution

To show that function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on $\mathbb{R}$, first, we need to show that $f'(x) > 0$.

Differentiating $f(x)$ w.r.t. x, we get

$$f'(x) = \frac{d}{dx}(x^3 - 3x^2 + 6x - 100)$$

$$f'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2) \quad (1)$$

Factorising Eq. (1): Add and subtract 1 from Eq. (1), we have

$$f'(x) = 3(x^2 - 2x + 2 + 1 - 1) = 3[(x^2 - 2x + 1) + (2 - 1)]$$

Therefore,

$$f'(x) = 3[(x-1)^2 + 1]$$

$$\Rightarrow f'(x) > 0$$

Hence, it is proved that the given function $f(x)$ is increasing function on $\mathbb{R}$.

9. The x-coordinate of a point on the line joining the points P(2, 2, 1) and Q(5, 1, –2) is 4. Find the z-coordinate.

Solution

According to section formula, if a point R(x, y, z) divides the line joining two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) in ratio a : b, then the coordinates of R(x, y, z) are given by

$$x = \frac{ax_2 + bx_1}{m+n}$$

$$y = \frac{ay_2 + by_1}{m+n}$$

$$z = \frac{az_2 + bz_1}{m+n}$$

Let the point on the line joining the points P(2, 2, 1) and Q(5, 1, –2) be A(x, y, z). Suppose that the point divides the line in a : 1 ratio. Then coordinates of point A are,

$$x = \frac{a(5) + 2}{a+1} \quad (1)$$

$$y = \frac{a(1) + 2}{a+1} \quad (2)$$

$$z = \frac{a(-2) + 1}{a+1} \quad (3)$$

It is given that $x = 4$; therefore, from Eq. (1), we have

$$\frac{a(5) + 2}{a+1} = 4$$

$$5a + 2 = 4(a + 1)$$

$$\Rightarrow 5a - 4a = 4 - 2$$

$$\Rightarrow a = 2$$

Substituting *a* = 2 in Eq. (3) to find the *z*-coordinate of point A, we get

$$z = \frac{(2)(-2) + 1}{2 + 1} = \frac{-4 + 1}{3}$$

That is, $z = -1$.

Thus, the *z*-coordinate of point on line joining points P(2, 2, 1) and Q(5, 1, –2) is –1.

10. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let *A* be the event "number obtained is even" and *B* be the event "number obtained is red". Find if *A* and *B* are independent events.

Solution

Let *S* = {1, 2, 3, 4, 5, 6} be the sample space of total number of events.

The total number of events in *S* is 6.

The probability that the number obtained in even is given as

$$P(A) = P(2, 4, 6) = \frac{3}{6} = \frac{1}{2}$$

The probability that the number obtained is red is given as

$$P(B) = P(1, 2, 3) = \frac{3}{6} = \frac{1}{2}$$

Two events are said to be independent if the probability of occurrence of one evet does not affect the probability of occurrence of the other event. Therefore, event *A* and event *B* are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

Now, the probability that the number obtained in "even and red" is given as

$$P(A \cap B) = P(2) = \frac{1}{6}$$

and $$P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(A \cap B) \neq P(A) \cdot P(B)$$

Hence, both event *A* and event *B* are not independent events.

11. Two tailors, A and B, earn Rs.300 and Rs.400 per day, respectively. A can stitch 6 shirts and 4 pair of trousers while B can stitch 10 shirts and 4 pair of trousers per day. To find how many days should each one of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP.

Solution

Suppose that tailor A works for x days and tailor B works for y days.

Since x and y are number of days,

$$x \geq 0, y \geq 0$$

Tailor A can stitch 6 shirts in a day and tailor B can stitch 10 shirts a day. It is desired to produce at least 60 shirts. So,

$$6x + 10y \geq 60$$

Also, tailor A can stitch 4 pair of trousers in a day and tailor B can stitch 4 pair of trousers in a day. It is desired to produce at least 32 pair of trousers. So,

$$4x + 4y \geq 32$$

It is given that tailor A earns Rs.300 and tailor B earns Rs.400 per day.

The total labour cost will be $(300x + 400y)$, which is to be minimize.

Formulation of LPP is the following:

Minimize objective function, $z = 300x + 400y$

Subject of constraints, $6x + 10y \geq 60$

$$4x + 4y \geq 32$$

$$x \geq 0$$

$$y \geq 0$$

12. Find: $\int \frac{dx}{5-8x-x^2}$

Solution

$$I = \int \frac{dx}{5 - 8x - x^2} = -\int \frac{dx}{x^2 + 8x - 5}$$

Making the denominator of integrand a perfect square, add and subtract 16 from the denominator of integrand, we get

$$I = -\int \frac{dx}{x^2 + 8x - 5 + 16 - 16} = -\int \frac{dx}{(x^2 + 8x + 16) + (-5 - 16)}$$

$$I = -\int \frac{dx}{(x+4)^2 - 21} = -\int \frac{dx}{(x+4)^2 - \left(\sqrt{21}\right)^2} \quad (1)$$

Let, $x + 4 = t$ (2)

Differentiating, we get

$$dx = dt \quad (3)$$

Substituting Eqs. (2) and (3) in Eq. (1), we get

$$I = -\int \frac{dt}{(t)^2 - \left(\sqrt{21}\right)^2} \quad (4)$$

Using standard integral, we get

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + C$$

Therefore,

$$I = -\int \frac{dt}{(t)^2 - \left(\sqrt{21}\right)^2} = -\frac{1}{2\sqrt{21}} \log\left(\frac{t-\sqrt{21}}{t+\sqrt{21}}\right) + C$$

Substituting $t = x + 4$ [from Eq. (2)], we get

$$I = -\frac{1}{2\sqrt{21}} \log\left(\frac{x+4-\sqrt{21}}{x+4+\sqrt{21}}\right) + C$$

$$\int \frac{dx}{5 - 8x - x^2} = -\frac{1}{2\sqrt{21}} \log\left(\frac{x+4-\sqrt{21}}{x+4+\sqrt{21}}\right) + C$$

This is the required solution.

SECTION C

13. If $\tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \frac{\pi}{4}$, then find the value of *x*.

Solution

Using trigonometric identity,

$$\tan^{-1} x_1 + \tan^{-1} x_2 = \tan^{-1}\left(\frac{x_1 + x_2}{1 - x_1 x_2}\right)$$

Therefore,

$$\tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \tan^{-1}\left(\frac{\frac{x-3}{x-4} + \frac{x+3}{x+4}}{1 - \frac{x-3}{x-4}\frac{x+3}{x+4}}\right)$$

$$\Rightarrow \tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \tan^{-1}\left(\frac{\frac{(x-3)(x+4)+(x+3)(x-4)}{(x-4)(x+4)}}{\frac{(x-4)(x+4)-(x-3)(x+3)}{(x-4)(x+4)}}\right)$$

$$\Rightarrow \tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4}$$

$$= \tan^{-1}\left(\frac{(x-3)(x+4)+(x+3)(x-4)}{(x-4)(x+4)} \times \frac{(x-4)(x+4)}{(x-4)(x+4)-(x-3)(x+3)}\right)$$

$$\Rightarrow \tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \tan^{-1}\left(\frac{(x-3)(x+4)+(x+3)(x-4)}{(x-4)(x+4)-(x-3)(x+3)}\right)$$

Using $a^2 - b^2 = (a-b)(a+b)$

$$\Rightarrow \tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \tan^{-1}\left(\frac{(x-3)(x+4)+(x+3)(x-4)}{(x^2-4^2)-(x^2-3^2)}\right)$$

Simplifying the numerator on RHS, we get

$$\tan^{-1}\frac{(x-3)(x+4)+(x+3)(x-4)}{(x^2-4^2)-(x^2-3^2)} = \tan^{-1}\left(\frac{x^2-3x+4x-12+x^2+3x-4x-12}{(x^2-4^2)-(x^2-3^2)}\right)$$

$$= \tan^{-1}\left(\frac{2x^2-24}{(x^2-4^2)-(x^2-3^2)}\right)$$

Simplifying the denominator on RHS, we get

$$\tan^{-1}\left(\frac{2x^2-24}{(x^2-4^2)-(x^2-3^2)}\right) = \tan^{-1}\left(\frac{2x^2-24}{x^2-4^2-x^2+3^2}\right)$$

$$= \tan^{-1}\left(\frac{2x^2-24}{x^2-16-x^2+9}\right)$$

$$= \tan^{-1}\left(\frac{2x^2-24}{-7}\right)$$

Therefore,

$$\tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \tan^{-1}\left(\frac{2x^2-24}{-7}\right)$$

We know that $\tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \frac{\pi}{4}$. Therefore,

$$\tan^{-1}\left(\frac{2x^2-24}{-7}\right) = \frac{\pi}{4}$$

$$\frac{2x^2-24}{-7} = \tan\left(\frac{\pi}{4}\right)$$

Substituting $\tan\left(\frac{\pi}{4}\right) = 1$, we get

$$\frac{2x^2-24}{-7} = 1$$

$$\Rightarrow \quad 2x^2 - 24 = -7$$

$$\Rightarrow \quad x^2 = \frac{-7+24}{2} = \frac{17}{2}$$

$$x = \pm\sqrt{\frac{17}{2}}$$

which is the required value of x.

14. Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

Solution

We will solve the above determinant using elementary row operations. Suppose

$$A = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Applying the matrix operation (elementary row operation) $R_1 \longrightarrow R_1 - R_2$, we get

$$A = \begin{vmatrix} a^2+2a-2a-1 & 2a+1-a-2 & 1-1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Use $a^2 - b^2 = (a-b)(a+b)$, we get

$$A = \begin{vmatrix} (a-1)(a+1) & a-1 & 0 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Take $(a-1)$ common from R_1

$$A = (a-1)\begin{vmatrix} a+1 & 1 & 0 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Applying elementary row operation $R_2 \longrightarrow R_2 - R_3$, we get

$$A = (a-1)\begin{vmatrix} a+1 & 1 & 0 \\ 2a+1-3 & a+2-3 & 1-1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)\begin{vmatrix} a+1 & 1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$A = (a-1)\begin{vmatrix} a+1 & 1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Take $(a-1)$ common from R_2:

$$A = (a-1)(a-1)\begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Expanding the determinant along C_3, we get

$A = (a-1)(a-1)\{0[6-3] - 0[3(a+1)-3] + 1[1(a+1)-2]\}$

$A = (a-1)(a-1)\{0-0+a-1\}$

$A = (a-1)(a-1)(a-1)$

$A = (a-1)^3$

Hence, it is proved.

OR

Find matrix A such that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$.

Solution:

We have the following:

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \text{ is a } 3 \times 2 \text{ matrix}$$

$$\begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix} \text{ is a } 3 \times 2 \text{ matrix}$$

Suppose A is a 2×2 matrix. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

Perform matrix multiplication

$$\begin{bmatrix} 2a - c & 2b - d \\ a & b \\ -3a + 4c & -3b + 4d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

Comparing elements of matrix on LHS to elements of matrix on RHS

$$2a - c = -1 \qquad (1)$$

$$2b - d = -8 \qquad (2)$$

$$a = 1 \qquad (3)$$

$$b = -2 \qquad (4)$$

Substituting $a = 1$ in Eq. (1), we get

$$2(1) - c = -1 \Rightarrow c = 2 + 1 \ \Rightarrow c = 3$$

Substituting $b = -2$ in Eq. (2), we get

$$2(-2) - d = -8 \Rightarrow d = -4 + 8 \ \Rightarrow d = 4$$

Therefore, matrix A is obtained as follows:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

15. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

Solution

Let

$$x^y = u \qquad (1)$$

$$y^x = v \qquad (2)$$

Therefore,

$$u + v = a^b$$

Differentiating w.r.t. *x*, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \tag{3}$$

Now, considering Eq. (1) and taking log on both sides, we get

$$\log\ x^y = \log\ u$$

$$\Rightarrow y \log x = \log u$$

Differentiating this equation, we get

$$dy \log x + y\frac{1}{x}dx = \frac{1}{u}du$$

Dividing both sides by *dx*, we get

$$\frac{dy}{dx}\log x + \frac{y}{x} = \frac{1}{u}\frac{du}{dx}$$

$$\Rightarrow \frac{du}{dx} = u\left(\log x\ \frac{dy}{dx} + \frac{y}{x}\right) \tag{4}$$

Now, considering Eq. (2) and taking log on both sides, we get

$$\log\ y^x = \log\ v$$

$$\Rightarrow x \log y = \log v$$

Differentiating this equation, we get

$$dx \log y + x\frac{1}{y}dy = \frac{1}{v}dv$$

Dividing both sides by *dx*, we get

$$\log y + \frac{x}{y}\frac{dy}{dx} = \frac{1}{v}\frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx} = v\left(\log y\ + \frac{x}{y}\frac{dy}{dx}\right) \tag{5}$$

Substituting Eqs. (4) and (5) in Eq. (3), we get

$$u\left(\log x\ \frac{dy}{dx} + \frac{y}{x}\right) + v\left(\log y\ + \frac{x}{y}\frac{dy}{dx}\right) = 0$$

Substituting the values of *u* and *v* from Eqs. (1) and (2), we get

$$x^y\left(\log x\ \frac{dy}{dx} + \frac{y}{x}\right) + y^x\left(\log y\ + \frac{x}{y}\frac{dy}{dx}\right) = 0$$

$$x^y \log x\ \frac{dy}{dx} + x^y\frac{y}{x} + y^x \log y\ + y^x\frac{x}{y}\frac{dy}{dx} = 0$$

Rearranging and collecting common terms together, we get

$$\left(x^y \log x + y^x \frac{x}{y}\right)\frac{dy}{dx} + x^y \frac{y}{x} + y^x \log y = 0$$

$$\left(x^y \log x + y^x \frac{x}{y}\right)\frac{dy}{dx} = -\left(x^y \frac{y}{x} + y^x \log y\right)$$

$$\frac{dy}{dx} = \frac{-\left(x^y \frac{y}{x} + y^x \log y\right)}{x^y \log x + y^x \frac{x}{y}}$$

$$\frac{dy}{dx} = \frac{-(x^{y-1}y + y^x \log y)}{x^y \log x + y^{x-1}x}$$

This is the required value of $\frac{dy}{dx}$.

OR

If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Solution

$$e^y(x+1) = 1$$

Taking log on both sides, we get

$$\log(e^y(x+1)) = \log 1$$

Use log(ab) = log$a$ + log$b$ and log1=0, we get

$$\log e^y + \log(x+1) = 0$$

$$y = -\log(x+1) \quad (1)$$

Differentiating Eq. (1) w.r.t. x, we get

Using differential formula $\frac{d}{dx}\log x = \frac{1}{x}$, we get

$$\frac{dy}{dx} = \frac{-1}{x+1} \quad (2)$$

Squaring both sides, we get

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{-1}{x+1}\right)^2$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{(x+1)^2} \quad (3)$$

Differentiating Eq. (2) again w.r.t. x, we get

Using differential formula $\frac{d}{dx}x^n = nx^{n-1}$, we get

$$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} \quad (4)$$

From Eqs. (1) and (2), we get

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

Hence, it is proved.

16. Find: $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)} d\theta$

Solution

Let

$$I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)} d\theta$$

$$I = \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4-4\cos^2\theta)} d\theta = \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4(1-\cos^2\theta))} d\theta$$

Using $\cos^2\theta + \sin^2\theta = 1$, we get

$$I = \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)} d\theta \quad (1)$$

Let $\sin\theta = t$ (2)

Differentiating Eq. (1), we get

$$\cos\theta\, d\theta = dt \quad (3)$$

Substituting Eqs. (2) and (3) in Eq. (1), we get

$$I = \int \frac{dt}{(4+t^2)(1+4t^2)} = \frac{1}{4}\int \frac{dt}{(4+t^2)\left(\frac{1}{4}+t^2\right)} \quad (4)$$

Integrating the above integral by partial fraction: If the denominator of integral has repeated quadratic term, then by partial fraction, we have

$$\frac{1}{(ax^2+bx+c)(px^2+qx+r)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(px^2+qx+r)}$$

Therefore,

$$\frac{1}{(4+t^2)(\frac{1}{4}+t^2)} = \frac{At+B}{(4+t^2)} + \frac{Ct+D}{(\frac{1}{4}+t^2)} \quad (5)$$

$$\Rightarrow \quad \frac{1}{(4+t^2)(\frac{1}{4}+t^2)} = \frac{(At+B)\left(\frac{1}{4}+t^2\right)+(Ct+D)(4+t^2)}{(4+t^2)(\frac{1}{4}+t^2)}$$

$$\Rightarrow 1 = \frac{1}{4}At + At^3 + \frac{1}{4}B + Bt^2 + 4Ct + 4D + Ct^3 + Dt^2$$

$$\Rightarrow 1 = (A+C)t^3 + (B+D)t^2 + \left(\frac{1}{4}A + 4C\right)t + \frac{1}{4}B + 4D \quad (6)$$

Since the coefficient of t^3 on LHS is zero, we put $A + C = 0 \Rightarrow A = -C$.

Also, the coefficient of t on LHS is zero and we put $\frac{1}{4}A + 4C = 0$

Substituting $A = -C$, we get

$$\frac{1}{4}(-C) + 4C = 0 \Rightarrow C = 0 \text{ and} \Rightarrow A = 0$$

Similarly, the coefficient of t^2 on LHS is zero and we put $B + D = 0 \Rightarrow B = -D$

Using Eq. (6), we get

$$\frac{1}{4}B + 4D = 1$$

Put $B = -D$, we get

$$\frac{1}{4}(-D) + 4D = 1 \Rightarrow 16D - D = 4 \Rightarrow D = \frac{4}{15}$$

Therefore,

$$B = -\frac{4}{15}$$

Put $A = 0\,, B = -\frac{4}{15}\,, C = 0\,, D = \frac{4}{15}$ in Eq. (5), we get

$$\frac{1}{(4+t^2)(\frac{1}{4}+t^2)} = \frac{-\frac{4}{15}}{(4+t^2)} + \frac{\frac{4}{15}}{(\frac{1}{4}+t^2)}$$

Substituting above relation in Eq. (4), the integral becomes

$$I = \frac{1}{4}\int \frac{dt}{(4+t^2)\left(\frac{1}{4}+t^2\right)} = \frac{1}{4}\int \frac{-\frac{4}{15}}{(4+t^2)} + \frac{\frac{4}{15}}{(\frac{1}{4}+t^2)}dt$$

$$= \frac{1}{4}\times\left(-\frac{4}{15}\right)\int \frac{1}{(4+t^2)}dt + \frac{1}{4}\times\left(\frac{4}{15}\right)\int \frac{1}{(\frac{1}{4}+t^2)}dt$$

$$= -\frac{1}{15}\int \frac{1}{(2^2+t^2)}dt + \frac{1}{15}\int \frac{1}{(\left(\frac{1}{2}\right)^2 + t^2)}dt$$

Use $\int \frac{1}{a^2+b^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$. Therefore,

$$I = -\frac{1}{15}\times\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + \frac{1}{15}\times\frac{1}{1/2}\tan^{-1}\left(\frac{t}{1/2}\right) + C$$

$$I = \frac{2}{15}\tan^{-1}(2t) - \frac{1}{30}\tan^{-1}\left(\frac{t}{2}\right) + C$$

From Eq. (2), put $t = \sin\theta$, we get

$$I = \frac{2}{15}\tan^{-1}(2\sin\theta) - \frac{1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right) + C$$

Therefore,

$$\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta = \frac{2}{15}\tan^{-1}(2\sin\theta) - \frac{1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right) + C$$

This is the required solution.

17. Evaluate: $\int_0^{\pi} \frac{x\,tan\,x}{sec\,x+tan\,x}dx$

Solution: Let,

$$I = \int_0^{\pi} \frac{x\,tan\,x}{sec\,x + tan\,x}dx \qquad (1)$$

Put $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$

Therefore,

$$I = \int_0^{\pi} \frac{x\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx = \int_0^{\pi} \frac{x\frac{\sin x}{\cos x}}{\frac{1+\sin x}{\cos x}}dx$$

$$I = \int_0^{\pi} \frac{x\sin x}{1+\sin x}dx \qquad (2)$$

Use property of integration:

$$\int_0^c f(x)\cdot dx = \int_0^c f(c-x)dx$$

Therefore,

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x}dx = \int_0^{\pi} \frac{\pi\sin x - x\sin x}{1+\sin x}dx$$

Therefore,

$$I = \int_0^{\pi} \pi\frac{\sin x}{1+\sin x}dx - \int_0^{\pi} \frac{x\sin x}{1+\sin x}dx$$

From Eq. (2), the second term on RHS is I:

$$I = \int_0^{\pi} \pi\frac{\sin x}{1+\sin x}dx - I$$

$$\Rightarrow 2I = \int_0^{\pi} \pi\frac{\sin x}{1+\sin x}dx$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi} \pi\frac{\sin x}{1+\sin x}dx$$

Divide and multiply integrand by $(1 - \sin x)$, we get

$$I = \frac{1}{2}\int_0^{\pi} \pi \frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$

Use $a^2 - b^2 = (a-b)(a+b)$

$$I = \frac{1}{2}\int_0^{\pi} \pi \frac{\sin x\,(1-\sin x)}{1^2 - \sin^2 x}\, dx$$

Using trigonometric identity, $\sin^2 x + \cos^2 x = 1$

$$I = \frac{1}{2}\int_0^{\pi} \pi \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx$$

Therefore,

$$I = \frac{1}{2}\int_0^{\pi} \pi \frac{\sin x}{\cos^2 x} dx - \frac{1}{2}\int_0^{\pi} \pi \frac{sin^2 x}{cos^2 x} dx$$

Put $\frac{\sin x}{\cos x} = \tan x$, we get

$$I = \frac{1}{2}\int_0^{\pi} \pi \frac{\tan x}{\cos x} dx - \frac{1}{2}\int_0^{\pi} \pi\, tan^2 x\, dx$$

Put $\frac{1}{\cos x} = \sec x$, we get

$$I = \frac{1}{2}\int_0^{\pi} \pi \tan x \sec x\, dx - \frac{1}{2}\int_0^{\pi} \pi \tan^2 x\, dx$$

Put $\tan^2 x = \sec^2 x - 1$:

$$I = \frac{1}{2}\int_0^{\pi} \pi \tan x \sec x\, dx - \frac{1}{2}\int_0^{\pi} \pi\,(\sec^2 x - 1)\, dx$$

$$I = \frac{1}{2}\int_0^{\pi} \pi \tan x \sec x\, dx - \frac{1}{2}\int_0^{\pi} \pi \sec^2 x\, dx + \frac{1}{2}\int_0^{\pi} \pi\, dx$$

Using standard trigonometric integrals, we get

$$\int \sec x \tan x\, dx = \sec x$$

$$\int sec^2 x\, dx = \tan x$$

and $\int dx = x$

$$I = \frac{1}{2}\pi[\sec x]_0^{\pi} - \frac{1}{2}\pi[\tan x]_0^{\pi} + \frac{1}{2}\pi[x]_0^{\pi}$$

$$I = \frac{1}{2}\pi[\sec \pi - \sec 0] - \frac{1}{2}\pi[\tan \pi - \tan 0] + \frac{1}{2}\pi[\pi - 0]$$

$$I = \frac{1}{2}\pi[-1-1] - \frac{1}{2}\pi[0-0] + \frac{1}{2}\pi[\pi - 0]$$

$$I = \frac{-2\pi}{2} - 0 + \frac{\pi^2}{2}$$

$$I = \frac{\pi}{2}(\pi - 2)$$

Therefore,

$$\int_0^{\pi} \frac{x \tan x}{sec\, x + \tan x} dx = \frac{\pi}{2}(\pi - 2)$$

This is the required value of integral.

OR

Evaluate: $\int_1^4 \{|x-1| + |x-2| + |x-4|\} dx$

Solution

Let

$$I = \int_1^4 \{|x-1| + |x-2| + |x-4|\} dx$$

$$I = \int_1^4 |x-1| dx + \int_1^4 |x-2| dx + \int_1^4 |x-4| dx$$

Now, by the definition of modulus of a function, we get

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

$$|x-1| = (x-1) \;\; 1 \leq x \leq 4$$

$$|x-2| = \begin{cases} -(x-2), & 1 \leq x \leq 2 \\ (x-2), & 2 \leq x \leq 4 \end{cases}$$

$$|x-4| = -(x-4), \quad 1 \leq x \leq 4$$

Therefore the integral becomes

$$I = \int_1^4 (x-1) dx - \int_1^2 (x-2) dx + \int_2^4 (x-2) dx - \int_1^4 (x-4) dx$$

Using standard integral $\int x^n dx = \frac{x^{n+1}}{n+1}$, we get

$$I = \left[\frac{x^2}{2} - x\right]_1^4 - \left[\frac{x^2}{2} - 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 - \left[\frac{x^2}{2} - 4x\right]_1^4$$

$$I = \left\{\left(\frac{4^2}{2} - 4\right) - \left(\frac{1^2}{2} - 1\right)\right\} - \left\{\left(\frac{2^2}{2} - 2 \times 2\right) - \left(\frac{1^2}{2} - 2\right)\right\} + \left\{\left(\frac{4^2}{2} - 2 \times 4\right) - \left(\frac{2^2}{2} - 2 \times 2\right)\right\} - \left\{\left(\frac{4^2}{2} - 4 \times 4\right) - \left(\frac{1^2}{2} - 4 \times 1\right)\right\}$$

$$I = \left\{\left(\frac{16}{2} - 4\right) - \left(\frac{1}{2} - 1\right)\right\} - \left\{\left(\frac{4}{2} - 4\right) - \left(\frac{1}{2} - 2\right)\right\} + \left\{\left(\frac{16}{2} - 8\right) - \left(\frac{4}{2} - 4\right)\right\} - \left\{\left(\frac{16}{2} - 16\right) - \left(\frac{1}{2} - 4\right)\right\}$$

$$I = \left\{\frac{16-8}{2} - \frac{1-2}{2}\right\} - \left\{\frac{4-8}{2} - \frac{1-4}{2}\right\} + \left\{\frac{16-16}{2} - \frac{4-8}{2}\right\} - \left\{\frac{16-32}{2} - \frac{1-8}{2}\right\}$$

$$I = \left\{\frac{8}{2} - \frac{-1}{2}\right\} - \left\{\frac{-4}{2} - \frac{-3}{2}\right\} + \left\{\frac{0}{2} - \frac{-4}{2}\right\} - \left\{\frac{-16}{2} - \frac{-7}{2}\right\}$$

$$I = \left\{\frac{8+1}{2}\right\} - \left\{\frac{-4+3}{2}\right\} + \left\{\frac{4}{2}\right\} - \left\{\frac{-16+7}{2}\right\}$$

$$I = \left\{\frac{9}{2}\right\} - \left\{\frac{-1}{2}\right\} + \left\{\frac{4}{2}\right\} - \left\{\frac{-9}{2}\right\} = \frac{9+1+4+9}{2}$$

Therefore, $I = \frac{23}{2}$. Hence,

$$\int_1^4 \{|x-1| + |x-2| + |x-4|\}dx = \frac{23}{2}$$

This is the required solution.

18. Solve the differential equation $(\tan^{-1} x - y)dx = (1 + x^2)dy$.

Solution:

$$(\tan^{-1} x - y)dx = (1 + x^2)dy$$

$$\frac{dy}{dx} = \frac{\tan^{-1} x - y}{1 + x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\tan^{-1} x}{1 + x^2} - \frac{y}{1 + x^2}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{1 + x^2} = \frac{\tan^{-1} x}{1 + x^2}$$

The above equation is of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \text{ with } P = \frac{1}{1 + x^2} \text{ and } Q = \frac{\tan^{-1} x}{1 + x^2}$$

To solve differential equation of this form, we find the integrating factor.

$$\text{I.F.} = e^{\int P(x)dx} = e^{\int \frac{1}{1+x^2}dx}$$

Using standard integral $\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$, we get

$$\text{I.F.} = e^{\tan^{-1}x}$$

Thus, the solution of integral is given as

$$y(x) \times (\text{I.F.}) = \int Q \times (\text{I.F.})\ dx$$

Therefore,

$$y(x) \times \left(e^{\tan^{-1}x}\right) = \int \frac{\tan^{-1}x}{1+x^2} \times \left(e^{\tan^{-1}x}\right)\ dx$$

Rearranging this equation, we get

$$y(x) \times \left(e^{\tan^{-1}x}\right) = \int \frac{dx}{1+x^2} \times \tan^{-1}x \times \left(e^{\tan^{-1}x}\right) \quad (1)$$

Substituting in Eq. (1), we get

$$\tan^{-1}x = t \quad (2)$$

Differentiating Eq. (2), we get

$$\frac{1}{1+x^2}dx = dt \quad (3)$$

$$\because \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

Substituting Eqs. (2) and (3) in Eq. (1), we get

$$y(x) \times (e^t) = \int t(e^t)\ dt \quad (4)$$

Solving above integral by parts, which states that

$$\int u(x)v(x)dx = u(x)\int v(x)dx - \int\left\{\frac{d}{dx}u(x) \times \int v(x)\right\} \quad (5)$$

Suppose in Eq. (4), $u(t) = t$ and $v(t) = e^t$, applying Eq. (5), we get

$$y(x) \times (e^t) = t\int e^t\ dt - \int \frac{d}{dt}(t) \times \int e^t dt$$

$$\Rightarrow \quad y(x) \times (e^t) = te^t - \int e^t dt + C \qquad \text{as} \int e^t dt = e^t + C$$

$$\Rightarrow \quad y(x) \times (e^t) = te^t - e^t + C = e^t(t-1) + C$$

$$\Rightarrow y(x) = \frac{e^t(t-1)+C}{e^t} = (t-1) + \frac{C}{e^t}$$

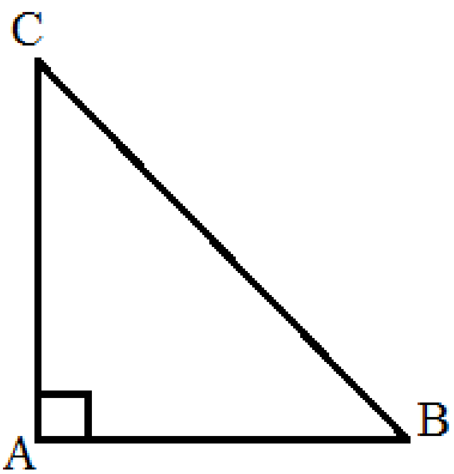
Substituting $t = \tan^{-1}x$ from Eq. (2), we get

$$y(x) = (\tan^{-1} x - 1) + \frac{C}{e^{\tan^{-1} x}}$$

19. Show that the points A, B, C with position vectors $2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$, and $3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$, respectively, are the vertices of a right-angled triangle. Hence, find the area of the triangle.

Solution

For vertices A, B, and C to form a right-angled triangle. Two vectors should be perpendicular to each other i.e. at an angle of 90°.



It is given that

$$\vec{A} = 2\hat{\imath} - \hat{\jmath} + \hat{k} \qquad (1)$$

$$\vec{B} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k} \qquad (2)$$

$$\vec{C} = 3\hat{\imath} - 4\hat{\jmath} - 4\hat{k} \qquad (3)$$

$\overrightarrow{AB}$ = Position Vector of B – Position Vector of A

$\overrightarrow{AB} = \vec{B} - \vec{A} = (\hat{\imath} - 3\hat{\jmath} - 5\hat{k}) - (2\hat{\imath} - \hat{\jmath} + \hat{k}) = -\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$

$\overrightarrow{BC}$ =Position Vector of C – Position Vector of B

$\overrightarrow{BC} = \vec{C} - \vec{B} = (3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}) - (\hat{\imath} - 3\hat{\jmath} - 5\hat{k}) = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

$\overrightarrow{CA}$ =Position Vector of A – Position Vector of C

$\overrightarrow{CA} = \vec{A} - \vec{C} = (2\hat{\imath} - \hat{\jmath} + \hat{k}) - (3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}) = -\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$

Now,

$\overrightarrow{AB} \cdot \overrightarrow{BC} = (-\hat{\imath} - 2\hat{\jmath} - 6\hat{k}) \cdot (2\hat{\imath} - \hat{\jmath} + \hat{k}) = -2 + 2 - 6 = -6$

$\overrightarrow{AB} \cdot \overrightarrow{CA} = (-\hat{\imath} - 2\hat{\jmath} - 6\hat{k}) \cdot (-\hat{\imath} + 3\hat{\jmath} + 5\hat{k}) = 1 - 6 - 30 = -35$

$\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{\imath} - \hat{\jmath} + \hat{k}) \cdot (-\hat{\imath} + 3\hat{\jmath} + 5\hat{k}) = -2 - 3 + 5 = 0$

Thus, $\overrightarrow{BC}\cdot\overrightarrow{CA}=0$ implies that vectors $\overrightarrow{BC}$ and $\overrightarrow{CA}$ are perpendicular to each other. Therefore ABC is a right-angled triangle.

Also if ABC is a right-angled triangle, Pythagoras theorem can be applied. According to Pythagoras theorem,

$a^2+b^2=c^2$

Where "c" is the hypotenuse and "a" and "b" are other sides of the triangle.

Here, $|A|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$

$|B|=\sqrt{1^2+(-3)^2+(-5)^2}=\sqrt{1+9+25}=\sqrt{36}$

$|C|=\sqrt{3^2+(-4)^2+(-4)^2}=\sqrt{9+16+16}=\sqrt{41}$

By Pythagoras theorem,

$$|A|^2+|B|^2=6+36=41=|C|^2$$

Therefore, ABC forms a right-angled triangle.

Now, the area of triangle, Area $=\frac{1}{2}\times$ Base $\times$ Height

Area $=\frac{1}{2}\times|\overrightarrow{AB}\times\overrightarrow{CA}|$

Now, $\overrightarrow{AB}\times\overrightarrow{CA}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -1 & -2 & -6\\ -1 & 3 & 5\end{vmatrix}$

$\overrightarrow{AB}\times\overrightarrow{CA}=\hat{\imath}[-10+18]-\hat{\jmath}[-5-6]+\hat{k}[-3-2]$

$\overrightarrow{AB}\times\overrightarrow{CA}=8\hat{\imath}+11\hat{\jmath}-5\hat{k}$

Therefore,

$$|\overrightarrow{AB}\times\overrightarrow{CA}|=\sqrt{8^2+11^2+(-5)^2}=\sqrt{64+121+25}=\sqrt{210}$$

Therefore,

Area $=\frac{1}{2}\times|\overrightarrow{AB}\times\overrightarrow{CA}|=\frac{1}{2}\sqrt{210}$

Area of triangle is $\frac{\sqrt{210}}{2}$.

20. Find the value of λ, if four points with position vectors $3\hat{\imath}+6\hat{\jmath}+9\hat{k}$, $\hat{\imath}+2\hat{\jmath}+3\hat{k}$, $2\hat{\imath}+3\hat{\jmath}+\hat{k}$ and $4\hat{\imath}+6\hat{\jmath}+\lambda\hat{k}$ are coplanar.

Solution

Let A, B, C, D be four points such that

$$A=3\hat{\imath}+6\hat{\jmath}+9\hat{k}$$

$$B=\hat{\imath}+2\hat{\jmath}+3\hat{k}$$

$$C = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$$

$$D = 4\hat{\imath} + 6\hat{\jmath} + \lambda\hat{k}$$

Then,

$$\overrightarrow{AB} = \text{Position Vector of } B - \text{Position Vector of } A$$

$$\overrightarrow{AB} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) - (3\hat{\imath} + 6\hat{\jmath} + 9\hat{k}) = -2\hat{\imath} - 4\hat{\jmath} - 6\hat{k}$$

$$\overrightarrow{AC} = \text{Position Vector of } C - \text{Position Vector of } A$$

$$\overrightarrow{AC} = (2\hat{\imath} + 3\hat{\jmath} + \hat{k}) - (3\hat{\imath} + 6\hat{\jmath} + 9\hat{k}) = -\hat{\imath} - 3\hat{\jmath} - 8\hat{k}$$

$$\overrightarrow{AD} = \text{Position Vector of } D - \text{Position Vector of } A$$

$$\overrightarrow{AC} = (4\hat{\imath} + 6\hat{\jmath} + \lambda\hat{k}) - (3\hat{\imath} + 6\hat{\jmath} + 9\hat{k}) = \hat{\imath} - 0\hat{\jmath} + (\lambda - 9)\hat{k}$$

It is given that these vectors are coplanar, therefore their scalar triple product should be zero.

$$\Rightarrow \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 0 \qquad (1)$$

Scalar triple product of vectors $\vec{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}, \vec{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}, \vec{C} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$ is given as

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Therefore,

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & \lambda - 9 \end{vmatrix}$$

Using Eq. (1), we get

$$\begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & \lambda - 9 \end{vmatrix} = 0$$

$$\Rightarrow -2(-3(\lambda - 9) - 0) + 4(-1(\lambda - 9) + 8) - 6(0 + 3) = 0$$

$$\Rightarrow -2(-3\lambda + 27) + 4(-\lambda + 9 + 8) - 6(3) = 0$$

$$\Rightarrow 6\lambda - 54 - 4\lambda + 68 - 18 = 0$$

$$\Rightarrow 2\lambda - 4 = 0$$

$$\Rightarrow \lambda = \frac{4}{2} = 2$$

The required value of $\lambda = 2$.

21. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X.

Solution

Let S be the sample space of events.

S = {{1,3},{1,5},{1,7},{3,1},{3,5}{3,7},{5,1}{5,3}{5,7},{7,1}{7,3}{7,5}}

The sample S contains 12 events.

If X denotes the sum of numbers on the two cards drawn.

$$X = \{4, 6, 8, 10, 12\}$$

Probability that sum of two drawn cards is 4, is probability that two cards drawn are {1, 3} or {3, 1}

$$P(\mathrm{X}=4) = P(1,3) \text{ or } P(3,1) = \frac{2}{12} = \frac{1}{6}$$

Probability that sum of two drawn cards is 6, is probability that two cards drawn are {1, 5} or {5, 1}

$$P(\mathrm{X}=6) = P(1,5) \text{ or } P(5,1) = \frac{2}{12} = \frac{1}{6}$$

Probability that sum of two drawn cards is 8 is probability that two cards drawn are {1, 7} or {7, 1} or {3, 5} or {5, 3}

$$P(\mathrm{X}=8) = P(1,7) \text{ or } P(7,1) \text{ or } P(3,5) \text{ or } P(5,3) = \frac{4}{12} = \frac{2}{6}$$

Probability that sum of two drawn cards is 10 is probability that two cards drawn are {3, 7} or {7, 3}

$$P(\mathrm{X}=10) = P(3,7) \text{ or } P(7,3) = \frac{2}{12} = \frac{1}{6}$$

Probability that sum of two drawn cards is 12 is probability that two cards drawn are {5, 7} or {7, 5}

$$P(\mathrm{X}=12) = P(5,7) \text{ or } P(7,5) = \frac{2}{12} = \frac{1}{6}$$

Mean is given as follows:

$$\text{Mean} = \sum P_i \mathrm{X}_i$$

Variance is given as follows:

$$\text{Variance} = \sum P_i {\mathrm{X}_i}^2 - \left(\sum P_i \mathrm{X}_i\right)^2$$

Now,

$$\sum P_i \mathrm{X}_i = P(\mathrm{X}=4)4 + P(\mathrm{X}=6)6 + P(\mathrm{X}=8)8 + P(\mathrm{X}=10)10 + P(\mathrm{X}=12)12$$

$$\Rightarrow \sum P_i \mathrm{X}_i = \frac{1}{6} \times 4 + \frac{1}{6} \times 6 + \frac{2}{6} \times 8 + \frac{1}{6} \times 10 + \frac{1}{6} \times 12$$

$$= \frac{1}{6}[4 + 6 + 16 + 10 + 12] = \frac{48}{6}$$

$$\sum P_i X_i = 8 \qquad (1)$$

and

$$\sum P_i {X_i}^2 = P(X = 4)4^2 + P(X = 6)6^2 + P(X = 8)8^2 + P(X = 10)10^2 + P(X = 12)12^2$$

$$\Rightarrow \sum P_i X_i = \frac{1}{6} \times 16 + \frac{1}{6} \times 36 + \frac{2}{6} \times 64 + \frac{1}{6} \times 100 + \frac{1}{6} \times 144$$

$$= \frac{1}{6}[16 + 36 + 128 + 100 + 144] = \frac{424}{6}$$

$$\sum P_i {X_i}^2 = 70.66 \qquad (2)$$

We know that, mean $= \sum P_i X_i = 8$

and variance $= \sum P_i {X_i}^2 - (\sum P_i X_i)^2 = 70.66 - 8^2 = 70.66 - 64 = 6.66$

Mean of X = 8 and variance of X = 6.66.

22. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer.

Solution

Let E_1 be the students who have 100 % attendance and E_2 be the students that are irregular.

It is given that 30 % students have 100% attendance

$$\Rightarrow \quad P(E_1) = 30\% = 0.3$$

It is given that 70 % students are irregular

$$\Rightarrow \quad P(E_2) = 70\% = 0.7$$

Let A be an event that student attains A grade.

Then, $\frac{A}{E_1}$ = student having A grade knowing that they have 100% attendance.

It is given that 70 % students who have 100 % attendance attain A grade.

$$\Rightarrow \quad P\left(\frac{A}{E_1}\right) = 70\ \% = 0.7$$

Similarly $\frac{A}{E_2}$ = student having A grade knowing that they are irregular.

It is given that 10 % of irregular students attain A grade.

$$\Rightarrow \quad P\left(\frac{A}{E_2}\right) = 10\ \% = 0.1$$

According of Bayes' Theorem:

A student chosen at random having grade A has 100 % attendance:

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \times P\left(\frac{A}{E_1}\right)}{P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)}$$

$$\Rightarrow \quad \frac{0.3 \ \times 0.7}{(0.3 \ \times 0.7) + (0.7 \ \times 0.1)} \Rightarrow \quad \frac{0.21}{0.21 + 0.07} = \frac{0.21}{0.28} = 3/4$$

Probability that a student chosen that has attained A grade has 100% attendance is 3/4.

Regularity is required not only in school but also in every aspect of life. As we can see in the above example that probability of success is ¾ in case of regularity and probability of success in case of irregularity is $\left(1 - \frac{3}{4}\right)$, that is, ¼. Similar is the case in all other aspects of life. Probability of success is always more for regularity.

23. Maximise $Z = x + 2y$ subject to the constraints

$$x + 2y \geq 100$$
$$2x - y \leq 0$$
$$2x + y \leq 200$$
$$x, y \geq 0$$

Solve the above LPP graphically.

Solution

Here the objective function and Constraints are given

Objective function: $Z = x + 2y$ (1)

Constraints: $x + 2y \geq 100$ (2)

$2x - y \leq 0$ (3)

$2x + y \leq 200$ (4)

$x, y \geq 0$ (5)

I. From the given constraints in linear inequalities, consider linear equations

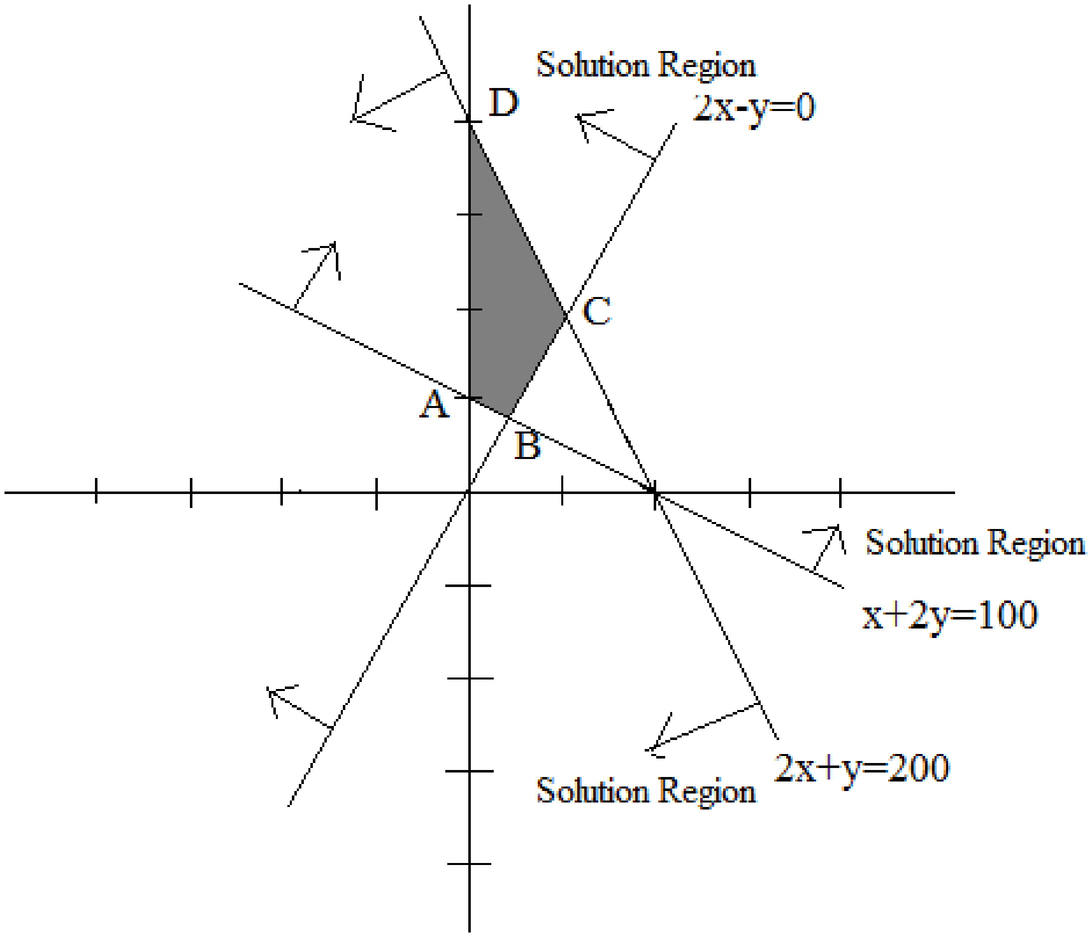
$x + 2y = 100$ (6)
$2x - y = 0$ (7)
$2x + y = 200$ (8)
$x, y = 0$ (9)

II. Plotting graph for each linear equation.

Consider Eq. (6), to find the point of intersection of equation at x–axis, put $y = 0 \Rightarrow x = 100$. To find the point of intersection at y–axis, put $x = 0 \Rightarrow y = 50$. (100, 0) and (0, 50) are points of intersection.

Consider Eq. (7), to find the point of intersection of equation at x–axis, put $y = 0 \Rightarrow x = 0$. To find the point of intersection at y–axis, put $x = 0 \Rightarrow y = 0$. (0, 0) is point of intersection. Some other points satisfying this equation are (50, 25), (100, 50).

Consider Eq. (8), to find the point of intersection of equation at x – axis, put $y = 0 \Rightarrow x = 100$. To find the point of intersection at y–axis, put $x = 0 \Rightarrow y = 200$. (100, 0) and (0, 200) are points of intersection.



III. Test the linear inequations to check the solution region. The test is as follows: put $x = 0$ and $y = 0$ in the inequality equation. If the equation is false the solution region is away from origin and if the equation is true solution region is towards origin.

From Eq. (2), for $x = 0$ and $y = 0$, $0 \geq 100$ false, so solution region is away from origin.

From Eq. (3), since the line is passing through origin, so consider a point on x–axis say (50, 0) so, $100 \leq 0$ False, so solution region will be away from positive x–axis.

From Eq. (4), for $x = 0$ and $y = 0$, $0 \leq 200$ true, so solution region is towards origin.

IV. The common solution regions of all linear inequations is the shaded portion. This is the feasible region.

Solving Eqs. (6) and (7), we have

$x + 2y = 100$

$2x - y = 0$

$\Rightarrow \quad x = 20, \qquad y = 40$

Solving Eqs. (7) and (8), we have

$2x + y = 200$

$2x - y = 0$

$\Rightarrow \quad x = 50, \qquad y = 100$

The corner points of feasible region are A (0, 50), B (20, 40), C (50, 100), D (0, 200).

V. Find the value of objective function at corner points of the feasible region.

From Eq. (1), objective function is $Z = x + 2y$

Corner Point	Objective Function
$A(0, 50)$	$Z = 0 + 2 \times 50 = 100$
$B(20, 40)$	$Z = 20 + 2 \times 40 = 100$
$C(50, 100)$	$Z = 50 + 2 \times 100 = 250$
$D(0, 200)$	$Z = 0 + 2 \times 200 = 400$ (maximum)

It is required to maximize the objective function. From the table the value of Z is maximum at point (0, 200). Hence the objective function has maximum value of 400 at point (0, 200).

SECTION –D

24. Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations $x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$.

Solution

Product of two matrices is given as follows,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) & (a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}) & (a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}) \\ (a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}) & (a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}) & (a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

Let A=$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and B=$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} -4(1)+4(1)+4(2) & -4(-1)+4(-2)+4(1) & -4(1)+4(-2)+4(3) \\ -7(1)+1(1)+3(2) & -7(-1)+1(-2)+3(1) & -7(1)+1(-2)+3(3) \\ 5(1)-3(1)-1(2) & 5(-1)-3(-2)-1(1) & 5(1)-3(-2)-1(3) \end{bmatrix}$$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is identity matrix I

$$AB = 8I \qquad (1)$$

Now we will solve the given system of equations, $x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$, that is, we need to find the value of x, y and z.

Writing the system of equations in matrix form, we get

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} \qquad (2)$$

However, $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = B$

Suppose $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X$ and $\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = C$

Therefore, Eq. (2) becomes, $BX = C$

$$\Rightarrow X = B^{-1}C \qquad (3)$$

Now from Eq. (1), $AB = 8I$

$$\Rightarrow A = 8I\, B^{-1}$$

$$\Rightarrow A = 8\, B^{-1} \qquad (\text{as } I\, B^{-1} = B^{-1})$$

$$\Rightarrow B^{-1} = \frac{A}{8} \qquad (4)$$

Substituting Eq. (4) in Eq. (3), we get

$$X = \frac{A}{8}C$$

Substituting the values of A, C and X in above equation, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4(4)+4(9)+4(1) \\ -7(4)+1(9)+3(1) \\ 5(4)-3(9)-1(1) \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

Therefore, the system of equations has solution $x = 3$, $y = –2$ and $z = –1$.

25. Consider $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \to \mathbb{R} - \left\{\frac{4}{3}\right\}$ given by $f(x) = \frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of f and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$.

Solution

It if given that

$$f(x) = \frac{4x+3}{3x+4} \text{ and } f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \to \mathbb{R} - \left\{\frac{4}{3}\right\}$$

For a function to be bijective, it should be injective as well as surjective, that is, one–one and onto.

Function f is injective, that is, one-to-one if for $x_1, x_2 \in \mathbb{R}$ such that if $f(x_1) = f(x_2)$ then, $x_1 = x_2$. So, let $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{4x_1+3}{3x_1+4} = \frac{4x_2+3}{3x_2+4}$$

$$\Rightarrow \quad (4x_1+3)(3x_2+4) = (4x_2+3)(3x_1+4)$$

$$\Rightarrow \quad 12\,x_1x_2 + 16x_1 + 9x_2 + 12 = 12\,x_1x_2 + 9x_1 + 16x_2 + 12$$

Cancelling common terms on both sides and rearranging the equation, we get

$$\Rightarrow \quad 16x_1 - 9x_1 = 16x_2 - 9x_2$$

$$\Rightarrow \quad x_1 = x_2$$

Therefore $f(x) = \frac{4x+3}{3x+4}$ is injective.

Now, a function f is surjective, that is, onto if, f has a domain X and codomain Y such that for $b \in Y$, there exists $a \in X$ such that $f(a) = b$. Suppose, $f(x) = y$.

$$\Rightarrow \quad \frac{4x+3}{3x+4} = y$$

$$\Rightarrow \quad 4x + 3 = y(3x+4)$$

$$\Rightarrow \quad 4x + 3 = 3xy + 4y$$

$$\Rightarrow \quad 4x - 3xy = 4y - 3$$

$$\Rightarrow \quad x(4-3y) = 4y - 3$$

$$\Rightarrow \quad x = \frac{4y-3}{4-3y} = g(y)$$

Since element of codomain Y has some element in domain X. Therefore $f(x)$ is surjective.

Since $f(x)$ is injective and surjective, therefore $f(x)$ is bijective.

The inverse of the function:

By definition, given the bijective function $f(x)$, such that $f : X \to Y$ and $f(x) = y$ where x ∈ X and y ∈ Y, the inverse function is defined as f^{-1} , such that f^{-1} : Y → X and f^{-1} (y) = x.

Therefore,

$$f^{-1}(x) = \frac{4x-3}{4-3x}$$

$$f^{-1}(0) = \frac{4\times 0-3}{4-3\times 0} = \frac{-3}{4}$$

$$f^{-1}(0) = \frac{-3}{4}$$

It is given that $f^{-1}(x) = 2$

$$\frac{4x-3}{4-3x} = 2$$

$$\Rightarrow \quad 4x-3 = 2(4-3x)$$

$$\Rightarrow \quad 4x-3 = 8-6x$$

$$\Rightarrow \quad 10x = 11$$

$$\Rightarrow \quad x = \frac{11}{10}$$

OR

Let $A = Q \times Q$ and let $*$ be a binary operation on A defined by $(a,b) * (c,d) = (ac, b+ad)$ for $(a,b),(c,d) \in A$. Determine, whether $*$ is commutative and associative. Then, with respect to * on A.

(i) Find the identity in A
(ii) Find the invertible elements of A

Solution

It is given that A is a set of rational number and for $(a,b),(c,d) \in A$,

$$(a,b) * (c,d) = (ac, b+ad) \qquad (1)$$

I. To check * is commutative, if $a,\ b \in A$, then

$$a * b = b * a$$

Consider, $a = (a_1,\ a_2), b = (\ b_1,\ b_2)$

$$(a_1,a_2) * (\ b_1,\ b_2) = (a_1b_1,\ a_2 + a_1\ b_2)$$
$$(\ b_1,\ b_2) * (a_1,a_2) = (b_1a_1,\ b_2 + b_1\ a_2)$$

$$\Rightarrow \quad (a_1, a_2) * (b_1, b_2) \neq (b_1, b_2) * (a_1, a_2)$$

Therefore * is not commutative.

II. To check * is associative, if $a,\ b,\ c \in A$, then

$$a * (b * c) = (a * b) * c$$

Consider, $a = (a_1,\ a_2), b = (b_1,\ \ b_2), c = (c_1,\ c_2)$

$$(a_1, a_2) * ((b_1,\ b_2) * (c_1,\ c_2)) = (a_1, a_2) * (b_1 c_1,\ b_2 + b_1\, c_2)$$
$$= a_1 b_1 c_1\,,\ \ a_2 + a_1(b_2 + b_1\, c_2)$$

Therefore,

$$(a_1, a_2) * ((b_1,\ b_2) * (c_1,\ c_2)) = a_1 b_1 c_1\,,\ \ a_2 + a_1\, b_2 + a_1 b_1\, c_2) \qquad (2)$$

$$\big((a_1, a_2) * (b_1,\ b_2)\big) * (c_1,\ c_2) = (a_1 b_1,\ a_2 + a_1\, b_2) * (c_1,\ c_2)$$

Therefore,

$$\big((a_1, a_2) * (b_1,\ b_2)\big) * (c_1,\ c_2) = (a_1 b_1 c_1\ , a_2 + a_1\, b_2 + a_1 b_1\, c_2) \qquad (3)$$

From Eqs. (2) and (3), we get

$$(a_1, a_2) * \big((b_1,\ b_2) * (c_1,\ c_2)\big) = \big((a_1, a_2) * (b_1,\ b_2)\big) * (c_1,\ c_2)$$

Therefore * is associative.

(i) Identity in *A*:

If $a,\ e \in A$, then, $a * e = a$

Consider, $a = (a_1,\ a_2),\ e = (e_1,\ \ e_2)$

$$(a_1, a_2) * (e_1,\ e_2) = (a_1 e_1,\ a_2 + a_1\, e_2) \qquad (4)$$

For $(a_1, a_2) * (e_1, e_2) = (a_1, a_2)$, the RHS of Eq. (4) should have $e_1 = 1$ and $e_2 = 0$

Therefore, $e = (1,\ 0)$ is the identity in *A*.

(ii) **Invertible elements in *A*:**

If $a,\ b,\ e \in A$, then, $a * b = e$

Consider, $a = (a_1,\ a_2),\ b = (b_1,\ \ b_2),\ e = (e_1,\ \ e_2)$

Therefore by relation,

$(a_1, a_2) * (b_1,\ b_2) = (e_1,\ e_2)$

We know that, $(e_1,\ e_2) = (1, 0)$

$(a_1, a_2) * (b_1,\ b_2) = (a_1 b_1,\ a_2 + a_1 b_2)$

$(a_1 b_1,\ a_2 + a_1 b_2) = (1, 0)$

Comparing the elements, we get

$$a_1 b_1 = 1 \quad \Rightarrow \quad b_1 = \frac{1}{a_1}$$

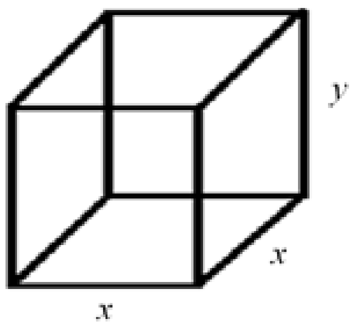
$$a_2 + a_1 b_2 = 0 \ \Rightarrow\ a_1 b_2 = -\ a_2 \quad \Rightarrow b_2 = -\frac{a_2}{a_1}$$

Therefore, , $b = \left(\frac{1}{a_1}\ ,\ -\frac{a_2}{a_1}\right)$ is an invertible element in *A*.

26. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

Solution

Let *x* be the length and *x* be the breadth and *y* be the height of the cuboid.



Let V be the volume of cuboid and S be the surface area of cuboid.

Volume of cuboid:

$$V = \text{Length} \times \text{Breadth} \times \text{Height}$$

$$\Rightarrow \quad V = x \times x \times y = x^2 y \qquad (1)$$

Surface area of cuboid:

$$S = 2\,(\text{Length} \times \text{Breadth} + \text{Length} \times \text{Height} + \text{Breadth} \times \text{Height})$$

$$\Rightarrow \quad S = 2\,(x \times x + x \times y + x \times y) = 2\,(x^2 + 2xy) \qquad (2)$$

From Eq. (1), substituting $y = \frac{V}{x^2}$ in Eq. (2), we get

$$S = 2\left(x^2 + 2x\frac{V}{x^2}\right) = 2\left(x^2 + \frac{2V}{x}\right) \qquad (3)$$

To check that surface area S is minimum, the derivative $\frac{d^2S}{dx^2} > 0$

Differentiating Eq. (3) w.r.t. x, we get

$$\frac{dS}{dx} = \frac{d}{dx}\left(2\left(x^2 + \frac{2V}{x}\right)\right) = 2\left(2x - \frac{2V}{x^2}\right) \qquad (4)$$

To find the critical point of minima put $\frac{dS}{dx} = 0$

$$\Rightarrow \quad 2\left(2x - \frac{2V}{x^2}\right) = 0$$

$$\Rightarrow \quad 2x = \frac{2V}{x^2} \quad \Rightarrow x^3 = V$$

$$\Rightarrow \quad x = V^{1/3} \qquad (5)$$

Differentiate Eq. (4) w.r.t x, we get

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left(2\left(2x - \frac{2V}{x^2}\right)\right) = 2\left(2 - 2V\frac{-2}{x^3}\right)$$

From Eq. (5), substituting $x^3 = V$, we get

$$\frac{d^2S}{dx^2} = 2\left(2 - 2V\frac{-2}{V}\right) = 2(2+4) = 12$$

Therefore, $\frac{d^2S}{dx^2} > 0 \Rightarrow x = V^{1/3}$ is point of minima.

From Eq. (1), we know that

$$y = \frac{V}{x^2}$$

Substituting $x = V^{1/3}$, we get

$$y = \frac{V}{\left(V^{\frac{1}{3}}\right)^2} = V^{\left(1-\frac{2}{3}\right)} = V^{\frac{1}{3}}$$

$\Longrightarrow \quad y = V^{\frac{1}{3}} = x$

$\Longrightarrow \quad x = y$

Therefore, Length = Breadth = Height. Hence, it is a cube.

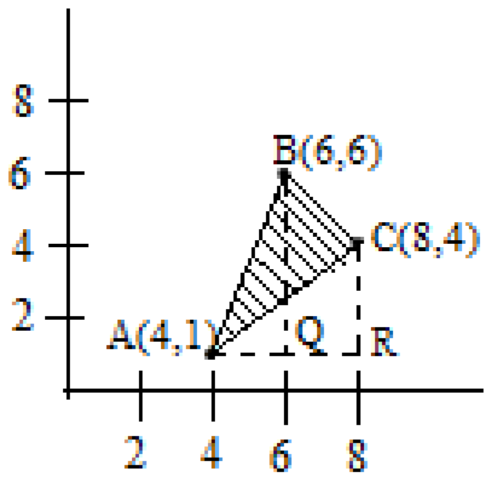
Hence, the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

27. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are A (4, 1), B (6, 6) and C (8, 4).

Solution

Let us plot the given vertices on graph as shown in the following figure. The shaded portion is the required area of triangle ABC.

Extend a line from point A parallel to *x*-axis. Draw a line from point B and a line from point C perpendicular to *x*-axis. Let the line from point B and C meet the line from point A at Q and R, respectively.



We know that equation of line segment for two points (x_1, y_1) and (x_2, y_2) is given as,

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

Equation of line segment AB with points (4, 1) and (6, 6) is

$$\frac{y-1}{6-1}=\frac{x-4}{6-4} \quad \Rightarrow \quad \frac{y-1}{5}=\frac{x-4}{2}$$

$\Rightarrow \quad 2(y-1)=5(x-4)$

$\Rightarrow \quad 2y-2=5x-20$

$\Rightarrow \quad 2y=5x-20+2$

$$\Rightarrow \quad y=\frac{5x-18}{2} \qquad (1)$$

Similarly, equation of line segment BC with points (6, 6) and (8, 4) is

$$\frac{y-6}{4-6}=\frac{x-6}{8-6} \quad \Rightarrow \quad \frac{y-6}{-2}=\frac{x-6}{2}$$

$\Rightarrow \quad 2(y-6)=-2(x-6)$

$\Rightarrow \quad 2y-12=-2x+12$

$\Rightarrow \quad 2y=-2x+12+12$

$$\Rightarrow \quad y=\frac{-2x+24}{2}=-x+12 \qquad (2)$$

Similarly, Equation of line segment CA with points (8, 4) and (4, 1) is

$$\frac{y-4}{1-4}=\frac{x-8}{4-8} \quad \Rightarrow \quad \frac{y-4}{-3}=\frac{x-8}{-4}$$

$\Rightarrow \quad -4(y-4)=-3(x-8)$

$\Rightarrow \quad -4y+16=-3x+24$

$\Rightarrow \quad -4y=-3x+24-16$

$$\Rightarrow \quad y=\frac{-3x+8}{-4}=\frac{3x-8}{4} \qquad (3)$$

Now from figure,

Area of ΔABC = area of $(ABQA)$ + area of $(BQRCB)$ − area of $(ACRA)$

$$= \int \text{equation of line segment AB} \,.\, dx$$

$$+ \int \text{equation of line segment BC} \,.\, dx$$

$$- \int \text{equation of line segment AC} \,.\, dx$$

Using Eqs. (1), (2) and (3), we get

$$\text{Area of } \Delta ABC = \int_4^6 \frac{5x-18}{2}dx + \int_6^8 (-x+12)dx - \int_4^8 \frac{3x-8}{4}dx$$

Using standard integral, $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\text{Area of } \Delta ABC = \frac{1}{2}\left[\frac{5x^2}{2} - 18x\right]_4^6 + \left[\frac{-x^2}{2} + 12x\right]_6^8 - \frac{1}{4}\left[\frac{3x^2}{2} - 8x\right]_4^8$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\left\{\left(\frac{5\times 6^2}{2} - 18\times 6\right) - \left(\frac{5\times 4^2}{2} - 18\times 4\right)\right\} + \left\{\left(\frac{-8^2}{2} + 12\times 8\right) - \left(\frac{-6^2}{2} + 12\times 6\right)\right\} - \frac{1}{4}\left\{\left(\frac{3\times 8^2}{2} - 8\times 8\right) - \left(\frac{3\times 4^2}{2} - 8\times 4\right)\right\}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\{(90-108)-(40-72)\} + \{(-32+96)-(-18+72)\} - \frac{1}{4}\{(96-64)-(24-32)\}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\{(-18)-(-32)\} + \{(64)-(54)\} - \frac{1}{4}\{(32)-(-8)\}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}(14) + (10) - \frac{1}{4}(40) = 7 + 10 - 10$$

$$\text{Area of } \Delta ABC = 7$$

OR

Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

Solution

Given: Equation of parabola is

$$4y = 3x^2 \qquad (1)$$

The parabola is along positive *y*–axis with vertex (0, 0).

Given: Equation of straight line is

$$3x - 2y + 12 = 0 \qquad (2)$$

To find the intersection points of line along *x*–axis put *y* = 0 in equation (2)

$\Rightarrow \quad 3x = -12 \quad \Rightarrow \quad x = -4$

The line intersects *x*-axis at point $(-4, 0)$.

To find the intersection points of line along *y*–axis put *x* = 0 in equation (2)

$\Rightarrow \quad -2y = -12 \quad \Rightarrow \quad y = 6$

The line intersects *y*-axis at point $(0, 6)$.

To find the points of intersection of line and parabola:

From Eq. (1), substituting $y = 3x^2/4$ in Eq. (2), we get

$$3x - 2\frac{3x^2}{4} + 12 = 0$$

$$\Rightarrow \quad -\frac{3}{2}x^2 + 3x + 12 = 0$$

$$\Rightarrow \quad \frac{-3x^2 + 6x + 24}{2} = 0$$

$$\Rightarrow \quad -3x^2 + 6x + 24 = 0$$

$$\Rightarrow \quad -3(x^2 - 2x - 8) = 0$$

$$\Rightarrow \quad x^2 - 2x - 8 = 0$$

$$\Rightarrow \quad x^2 - 4x + 2x - 8 = 0$$

$$\Rightarrow \quad x(x-4) + 2(x-4) = 0$$

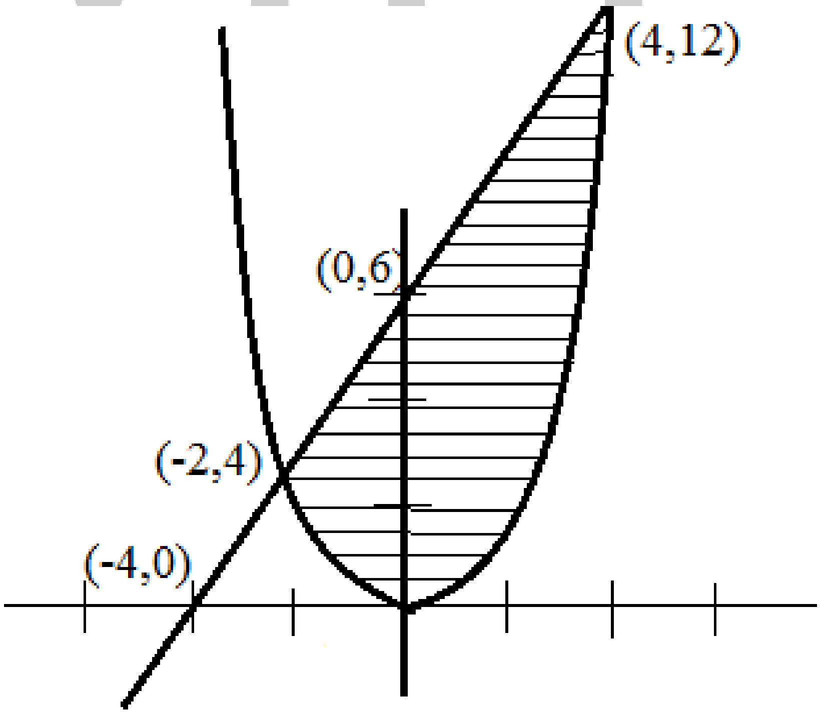
$$\Rightarrow \quad (x+2)(x-4) = 0$$

$$\Rightarrow \quad x = -2, 4$$

Substituting $x = -2$ in Eq. (1), we get $4y = 3(-2)^2 \Rightarrow y = 3$

Substituting $x = 4$ in Eq. (1), we get $4y = 3(4)^2 \Rightarrow y = 12$

Therefore, (4, 12) and (–2, 3) are points of intersection of line and parabola.



The area enclosed is the shaded portion.

Area = Area under straight line – Area under parabola

From Eq. (1): $y_1 = 3x^2/4$

From Eq. (2): $y_2 = (3x + 12)/2$

$$\text{Area} = \int_{-2}^{4} y_2 dx - \int_{-2}^{4} y_1 dx$$

$$\text{Area} = \int_{-2}^{4} \frac{3x + 12}{2} dx - \int_{-2}^{4} \frac{3x^2}{4} dx$$

Using standard integral, $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\text{Area} = \frac{1}{2}\left[\frac{3x^2}{2} + 12x\right]_{-2}^{4} - \frac{1}{4}\left[\frac{3x^3}{3}\right]_{-2}^{4}$$

$$\text{Area} = \frac{1}{2}\left\{\left(\frac{3 \times 4^2}{2} + 12 \times 4\right) - \left(\frac{3 \times (-2)^2}{2} + 12 \times (-2)\right)\right\} - \frac{1}{4}\{(4)^3 - (-2)^3\}$$

$$\text{Area} = \frac{1}{2}\{(24 + 48) - (6 - 24)\} - \frac{1}{4}\{64 + 8\}$$

$$\text{Area} = \frac{1}{2}\{72 + 18\} - \frac{1}{4}\{64 + 8\}$$

$$\text{Area} = \frac{1}{2}\{72 + 18\} - \frac{1}{4}\{64 + 8\}$$

$$\text{Area} = 45 - 18 = 27$$

The area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$ is 27.

28. Find the particular solution of the differential equation $(x - y)\frac{dy}{dx} = (x + 2y)$, given that $y = 0$ when $x = 1$.

Solution

$$(x - y)\frac{dy}{dx} = (x + 2y)$$

$$\frac{dy}{dx} = \frac{(x + 2y)}{(x - y)} \qquad (1)$$

$$\text{Let } y = vx \qquad (2)$$

Differentiate Eq. (2) w.r.t *x*

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (3)$$

Substituting Eqs. (2) and (3) in Eq. (1) , we get

$$v + x\frac{dv}{dx} = \frac{(x + 2vx)}{(x - vx)} \quad \Rightarrow v + x\frac{dv}{dx} = \frac{x(1 + 2v)}{x(1 - v)}$$

$$v + x\frac{dv}{dx} = \frac{(1+2v)}{(1-v)}$$

$$x\frac{dv}{dx} = \frac{(1+2v)}{(1-v)} - v \qquad \Rightarrow x\frac{dv}{dx} = \frac{(1+2v) - v(1-v)}{(1-v)}$$

$$x\frac{dv}{dx} = \frac{1+2v-v+v^2}{(1-v)}$$

$$x\frac{dv}{dx} = \frac{1+v+v^2}{(1-v)}$$

Rearranging above equation, we get

$$\frac{1-v}{1+v+v^2}dv = \frac{dx}{x}$$

Integrating, we get

$$\int \frac{1-v}{1+v+v^2}\ dv = \int \frac{dx}{x}$$

$$-\int \frac{v-1}{1+v+v^2}\ dv = \int \frac{dx}{x}$$

Rewriting the numerator: $v - 1 = \frac{1}{2}(2v+1) - \frac{3}{2}$

$$-\int \frac{\frac{1}{2}(2v+1) - \frac{3}{2}}{1+v+v^2}\ dv = \int \frac{dx}{x}$$

$$-\frac{1}{2}\int \frac{(2v+1)}{1+v+v^2}\ dv\ + \frac{3}{2}\int \frac{1}{1+v+v^2}\ dv = \int \frac{dx}{x}$$

Now, we know that the integral on RHS is log x + C, where C is constant of integration. Therefore,

$$-\frac{1}{2}\int \frac{(2v+1)}{1+v+v^2}\ dv\ + \frac{3}{2}\int \frac{1}{1+v+v^2}\ dv = \log x + \ C_1 \qquad (4)$$

Consider the integral:

$$\int \frac{(2v+1)}{1+v+v^2}\ dv$$

Substituting $1 + v + v^2 = t$

Differentiate it, we get $(2v+1)dv = dt$

Substituting these in the above integral, we get

$$\int \frac{(2v+1)}{1+v+v^2}\ dv = \int \frac{dt}{t} = \log t + C_2$$

Substituting the value of t, we get

$$\int \frac{(2v+1)}{1+v+v^2}\ dv = \log(1+v+v^2) + C_2$$

Substituting the value of above integral in Eq. (4), we get

$$-\frac{1}{2}\log(1+v+v^2) + C_2 + \frac{3}{2}\int \frac{1}{1+v+v^2}\ dv = \log x + C_1$$

$$-\frac{1}{2}\log(1+v+v^2) + \frac{3}{2}\int \frac{1}{1+v+v^2}\ d$$
$$= \log x + C_1 - C_2 \qquad (5)$$

Now, consider the integral:

$$\int \frac{1}{1+v+v^2}\ dv$$

Making perfect square, adding and subtracting ¼ in the above equation, we get

$$\int \frac{1}{1+v+v^2-\frac{1}{4}+\frac{1}{4}}\ dv = \int \frac{1}{(v^2+v+\frac{1}{4})+(1-\frac{1}{4})} dv$$

$$= \int \frac{1}{\left(v+\frac{1}{2}\right)^2+\frac{3}{4}} dv$$

$$= \int \frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} dv$$

Use standard integral $\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$:

$$\int \frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\ dv = \frac{1}{\sqrt{3}/2}\tan^{-1}\frac{v+1/2}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{(2v+1)/2}{\sqrt{3}/2}$$

Therefore,

$$\int \frac{1}{1+v+v^2}\ dv = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C_3$$

Substitute the value of integral in Eq. (5), we get

$$-\frac{1}{2}\log(1+v+v^2) + \frac{3}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C_3 = \log x + C_1 - C_2$$

$$-\frac{1}{2}\log(1+v+v^2) + \frac{3}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = \log x + C_1 - C_2 - C_3$$

Using Eq. (2), substituting the value of $v = \frac{y}{x}$ and since C_1, C_2, C_3 are constants of integration, let $C_1 - C_2 - C_3 = C$

$$\Rightarrow \quad -\frac{1}{2}\log\left(1+\frac{y}{x}+\left(\frac{y}{x}\right)^2\right)+\sqrt{3}\tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right)=\log x + C$$

It is given that $y = 0$ when $x = 1$

$$\Rightarrow \quad -\frac{1}{2}\log(1)+\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\log 1 + C$$

$$\Rightarrow \quad C=\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

We know that

$$\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad C=\sqrt{3}\frac{\pi}{6}=\frac{\pi}{2\sqrt{3}}$$

$$\Rightarrow \quad -\frac{1}{2}\log\left(1+\frac{y}{x}+\left(\frac{y}{x}\right)^2\right)+\sqrt{3}\tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right)=\log x + \frac{\pi}{2\sqrt{3}}$$

Simplifying the equation, we get

$$-\frac{1}{2}\log\left(\frac{x+yx+y^2}{x}\right)+\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}\,x}\right)=\log x+\frac{\pi}{2\sqrt{3}}$$

This is the particular solution of the given differential equation.

29. Find the coordinates of the point where the line through the points (3, –4, –5) and (2, –3, 1) crosses the plane determined by the points (1, 2, 3), (4, 2, –3) and (0, 4, 3).

Solution

Equation of a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=s$$

Therefore, equation of line passing through points (3, –4, –5) and (2, –3, 1) is

$$\frac{x-3}{2-3}=\frac{y-(-4)}{(-3)-(-4)}=\frac{z-(-5)}{1-(-5)}=s$$

$$\Rightarrow \quad \frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=s$$

$$\Rightarrow \quad \frac{x-3}{-1}=s \quad \Rightarrow \quad x=-s+3 \qquad (1)$$

$$\frac{y+4}{1}=s \quad \Rightarrow \quad y=s-4 \qquad (2)$$

$$\frac{z+5}{6}=s \quad \Rightarrow \quad z=6s-5 \qquad (3)$$

Suppose that $(-s+3, s-4, 6s-5)$ be the coordinates of the point of intersection.

Consider that the equation of plane is passing through (1, 2, 3) is given by

$$n(x-1)+l(y-2)+m(z-3)=0 \qquad (4)$$

Where (n, l, m) are ratios of direction of the normal to the plane.

The plane passes through point (4, 2, –3); therefore, it must satisfy equation of plane given by Eq. (4).

Substituting $x = 4$, $y = 2$, $z = -3$ in Eq. (1), we get

$$n(4-1)+l(2-2)+m(-3-3)=0$$

$$3n+0l-6m=0 \qquad (5)$$

The plane also passes through point (0, 4, 3); therefore, it must satisfy equation of plane given by Eq. (4).

Substituting $x = 0$, $y = 4$, $z = 3$ in Eq. (1), we get

$$n(0-1)+l(4-2)+m(3-3)=0$$

$$-n+2l+0m=0 \qquad (6)$$

Solving Eqs. (5) and (6) by cross multiplication method, we get

$$\frac{n}{0\times 0-(2)\times(-6)}=\frac{-l}{0\times 3-(-6)\times(-1)}=\frac{m}{2\times 3-0\times(-1)}$$

$$\Rightarrow \quad \frac{n}{12}=\frac{-l}{-6}=\frac{m}{6}$$

Suppose $\frac{n}{12}=\frac{l}{6}=\frac{m}{6}=p$

$$\Rightarrow \quad n=12p, \quad l=6p \quad ,m=6p$$

Substituting these values in Eq. (4), we get

$$12p(x-1)+(6p)(y-2)+6p(z-3)=0$$
$$\Rightarrow \quad p[12(x-1)+6(y-2)+6(z-3)]=0$$
$$\Rightarrow \quad 12x-12+6y-12+6z-18=0$$

$$\Rightarrow \quad 12x+6y+6z-42=0$$

Substituting values of x, y, and z from Eqs. (1), (2), and (3), we get

$12(-s+3)+6(s-4)+6(6s-5)-42=0$

$\Rightarrow \quad -12s+36+6s-24+36s-30-42=0$

$\Rightarrow \quad 30s-60=0$

$\Rightarrow \quad s=\frac{60}{30}=2$

Substituting values of in Eqs. (1), (2) and (3) to get the corresponding values of x, y and z:

$$x = -s + 3 = -2 + 3 = 1$$

$$y = s - 4 = 2 - 4 = -2$$

$$z = 6s - 5 = 6 \times 2 - 5 = 7$$

Thus, the coordinates of point of intersection are $(1, -2, 7)$.

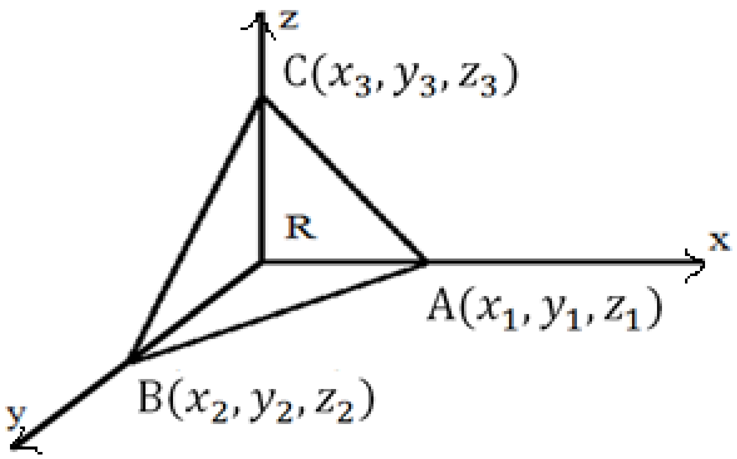
OR

A variable plane which remains at a constant distance $3p$ from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of the triangle ABC is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$$

Solution

It is given that the distance of variable plane from the origin is $3p$.



Let the variable plane be given as

$$a_1 x + a_2 y + a_3 z = 3p \qquad (1)$$

where a_1, a_2, a_3 are direction cosines of normal to the plane.

Let the coordinates of points cuts the axes at $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$

Now, the plane meets x-axis at point A. So, put $y = 0$ and $z = 0$ in Eq. (1), we get

$$a_1 x = 3p \quad \Rightarrow x = \frac{3p}{a_1}$$

Therefore, the coordinates to point A are $(\frac{3p}{a_1}, 0, 0)$. (2)

Similarly, the plane meets y-axis at point B. So, put x = 0 and z = 0 in Eq. (1) , we get

$$a_2 y = 3p \quad \Rightarrow y = \frac{3p}{a_2}$$

Therefore, the coordinates to point B are

$$(0, \frac{3p}{a_2}, 0) \qquad (3)$$

Similarly, the plane meets *z*-axis at point C. So, put *x* = 0 and *y* = 0 in Eq. (1) , we get

$$a_3 z = 3p \quad \Rightarrow z = \frac{3p}{a_3}$$

Therefore, the coordinates to point C are

$$(0, 0, \frac{3p}{a_3}) \qquad (4)$$

Suppose, that the centroid of triangle ABC is R (*x*, *y*, *z*). Coordinates of centroid are given as,

$$(x, y\ z) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right) \qquad (5)$$

From the coordinates of A, B, C found in Eqs. (2), (3) and (4), we get

$$x_1 = \frac{3p}{a_1}, \qquad x_2 = 0, \qquad x_3 = 0$$

$$y_1 = 0, \qquad y_2 = \frac{3p}{a_2}, \qquad y_3 = 0$$

$$z_1 = 0, \qquad z_2 = 0, \qquad z_3 = \frac{3p}{a_3}$$

Substituting all these values in Eq. (5), we get

$$(x, y\ z) = \left(\frac{\frac{3p}{a_1} + 0 + 0}{3}, \frac{0 + \frac{3p}{a_2} + 0}{3}, \frac{0 + 0 + \frac{3p}{a_3}}{3}\right) = \left(\frac{p}{a_1}, \frac{p}{a_2}, \frac{p}{a_3}\right)$$

Therefore,

$$x = \frac{p}{a_1} \quad \Rightarrow a_1 = \frac{p}{x} \qquad (6)$$

$$y = \frac{p}{a_2} \quad \Rightarrow a_2 = \frac{p}{y} \qquad (7)$$

$$z = \frac{p}{a_3} \quad \Rightarrow a_2 = \frac{p}{z} \qquad (8)$$

We know that a_1, a_2, a_3 are the direction of cosines of normal to the plane. Therefore,

$$a_1{}^2 + a_2{}^2 + a_3{}^2 = 1$$

Substituting the values of a_1, a_2, a_3 from Eqs. (6), (7) and (8), we get

$$\left(\frac{p}{x}\right)^2 + \left(\frac{p}{y}\right)^2 + \left(\frac{p}{z}\right)^2 = 1$$

$$p^2\left[\left(\frac{1}{x}\right)^2+\left(\frac{1}{y}\right)^2+\left(\frac{1}{z}\right)^2\right]=1$$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$$

Hence, it is proved.

