CBSE 2018 Solved Paper – Mathematics

1. If $a * b$ denotes the larger of a and b and if $a \circ b = (a * b) + 3$, then write the value of $(5) \circ (10)$, where * and $\circ$ are binary operations.

Solution

Here, $a * b$ denotes the larger of a and b. Now,

$$a \circ b = (a * b) + 3$$

To find $(5) \circ (10)$:

$$a \circ b = (a * b) + 3$$
$$(5) \circ (10) = (5 * 10) + 3$$
$$= 10 + 3 = 53$$

2. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$.

Solution

The magnitude of two vectors $\vec{a}$ and $\vec{b}$ is the same; therefore,

$$|\vec{a}| = |\vec{b}|;\ \theta = 60°,\ \vec{a} \cdot \vec{b} = \frac{9}{2}$$

We know that

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

To find $|\vec{a}|, |\vec{b}|$:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\frac{9}{2} = |\vec{a}|^2 \times \frac{1}{2} \qquad \left(\text{as } \cos 60° = \frac{1}{2};\ |\vec{a}| = |\vec{b}|\right)$$

$$\frac{9}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

That is, $|\vec{a}|^2 = 9$; $|\vec{a}| = \pm 3$.

Since the magnitude of a vector is not negative, $|\vec{a}| = 3$. Therefore,

$$|\vec{a}| = |\vec{b}| = 3 \qquad \text{(as it is given that } |\vec{a}| = |\vec{b}|\text{)}$$

3. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew symmetric, find the values of a and b.

Solution

Since matrix A is of skew symmetric, we have

$$A^{T} = -A$$

$$A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$$

$$A^{T} = \begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$$

$$-A = \begin{bmatrix} 0 & -a & 3 \\ -2 & 0 & 1 \\ -b & -1 & 0 \end{bmatrix}$$

Therefore, $a = -2$ and $b = 3$.

4. Find the value of $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$.

Solution

To find $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$: We have

$$\tan^{-1}\sqrt{3} = \frac{\pi}{3} \qquad (1)$$

$$\cot^{-1}\sqrt{3} = +5\frac{\pi}{6}$$

$$\cot^{-1}(\sqrt{-3}) = \left(+5\frac{\pi}{6}\right) \qquad (2)$$

Substituting Eqs. (1) and (2) in the given equation, we get

$$\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \left(5\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} - 5\frac{\pi}{6}$$

$$\Rightarrow \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6}$$

Therefore, $$\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = -\frac{\pi}{2}$$

5. The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost, we mean the instantaneous rate of change of total cost at any level of output.

Solution

The marginal cost is the rate of change of total cost w.r.t. the output.

Let MC be marginal cost; therefore,

$$\text{MC} = \frac{dC}{dx}$$

It is given that

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

We need to find marginal cost when three units are produced, that is,

$$\text{MC} = \left.\frac{dC}{dx}\right|_{x=3}$$

Now,
$$\text{MC} = \frac{d(C)}{dx}$$

$$\text{MC} = \frac{d(0.005x^3 - 0.02x^2 + 30x + 5000)}{dx}$$

$$\text{MC} = \frac{d(0.005x^3)}{dx} - \frac{d(0.02x^2)}{dx} + \frac{d(30x)}{dx} + \frac{d(5000)}{dt}$$

$$\text{MC} = 3 \times 0.005x^2 - 2 \times 0.02x + 30 + 0$$

$$\text{MC} = 0.015x^2 - 0.04x + 30$$

We need to find MC at $x = 3$: Substituting $x = 3$, we get

$$\text{MC} = 0.015(3)^2 - 0.04(3) + 30 = 30.015$$

Hence, the required marginal cost is approximately Rs.30.02.

6. Differentiate: $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

Solution

Differentiating the given $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ w.r.t. x, we get

$$\frac{d}{dx}\left[\tan^{-1}\frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}\right]$$

$$\Rightarrow \frac{d}{dx}\left(\tan^{-1}\cot\frac{x}{2}\right)$$

$$\Rightarrow \frac{d}{dx} = \frac{\pi}{2} - \frac{x}{2}$$

That is,
$$\frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}$$

7. Given: $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

Solution

Given: $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

To prove that $2A^{-1} = 9I - A$:

$$A^{-1} = \frac{1}{(2\times 7)-(12)}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & \frac{3}{2} \\ 2 & 1 \end{bmatrix}$$

Now, $2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \equiv \text{LHS}$

Also, $9I = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$

$\Rightarrow 9I - \text{A}$

$\Rightarrow \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \equiv \text{RHS}$

Therefore, LHS = RHS.

Hence, the given equation is proved.

8. Prove that $3\sin^{-1} x = \sin^{-1}(3x - 4x^3),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Solution

Given: $x \in \left(\frac{-1}{2}, \frac{1}{2}\right)$

To prove that

$$3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$$

Substituting $x = \sin\theta$ in RHS of the above equation, we get

$$\sin^{-1}[3\sin\theta - 4(\sin\theta)^3]$$

$\Rightarrow \sin^{-1} + (3\sin\theta - 4\sin^3\theta)$

$\Rightarrow \sin^{-1}(\sin 3\theta)$ (as $\sin 3x = 3\sin x - 4\sin^3 x$)

$\Rightarrow 3\theta$ [as $\sin^{-1}(\sin x) = x$]

Now, $x = \sin\theta$

$\sin^{-1}(x) = \theta$

$= 3\sin^{-1}(x)$

Therefore, LHS = RHS.

Hence, the given equation is proved.

9. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution

We need to find the probability of obtaining the sum 8; given that the red die resulted in a number less than 4.

- F: number on red die is less than 4
- E: sum of numbers is 8

We need to find $P(E/F)$:

$$E = \{(2, 6), (3, 5), (4, 4), (5, 6), (6, 2)\}$$

$$P(E) = \frac{5}{36}$$

Also,

$$F = \{(1, 1), (2, 1), \ldots (6, 1), (1, 2), (2, 2), \ldots, (6, 2), (1, 3), (2, 3), \ldots, (6, 3)\}$$

$$P(F) = \frac{18}{36}$$

Also, $E \cap F = \{(5,3), (6,2)\}$; therefore,

$$P(E \cap F) = \frac{2}{36}$$

$$P\left(\frac{E}{F}\right) = \frac{p(E \cap F)}{P(F)}$$

$$= \frac{2/36}{18/36} = \frac{2}{18} = \frac{1}{9}$$

Therefore, the required probability is $\frac{1}{9}$.

10. If θ is the angle between two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$, find $\sin\theta$.

Solution

We have

$$\vec{a} = i - 2j + 3k$$

and $$\vec{b} = 3i - 2j + k$$

$$\vec{a} \cdot \vec{b} = (i - 2j + 3k)(3i - 2j + k)$$

$$= (3 \times 1 + (-2)(-2) + (3 \times 1))$$

$$= (3 + 4 + 3)$$

$$= 10$$

The magnitude of vectors is given by

$$|a| = \sqrt{1+4+9} = \sqrt{14}$$

$$|b| = \sqrt{9+4+1} = \sqrt{14}$$

Now, $$\cos\theta = \frac{a \cdot b}{|a||b|}$$

The angle between two vectors is given by

$$\theta = \cos^{-1}\left(\frac{a.b}{|a||b|}\right)$$

$$= \cos^{-1}\left[\frac{10}{(\sqrt{14})(\sqrt{14})}\right]$$

$$= \cos^{-1}\frac{10}{14}$$

$$= \cos^{-1}\frac{5}{7}$$

$$= \cos^{-1}(0.795)$$

$$= 45.5°$$

Now, $$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\frac{25}{49}}$$

$$= \sqrt{\frac{49-25}{49}} = \sqrt{\frac{24}{7}}$$

That is, $\sin\theta = \frac{2\sqrt{6}}{7}; \cos\theta = \frac{10}{14}$.

11. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants.

Solution

We have

$$y = ae^{bx+5}$$

Taking log on both sides of this equation, we get

$$\log y = \log(ae^{bx+5})$$

$$\log y = \log + \log(e^{bx+5})$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = 0 + \frac{d}{dx}(bx+5)$$

$$\frac{1}{y}\frac{dy}{dx} = b$$

$$\frac{1}{y}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\frac{d^2y}{dx^2} + y\left(\frac{dy}{dx}\right)^2 = 0$$

which is the required differential equation representing the given family of curves.

12. Evaluate: $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$

Solution

We have

$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx = \int \frac{2\cos^2 x - 1 + 2(1-\cos^2 x)}{\cos^2 x}dx$$

$$= \int \left(2 - \frac{1}{\cos^2 x} + \frac{2}{\cos^2 x} - 1 \right) dx$$

$$= \int \left(1 + \frac{1}{\cos^2 x} \right) dx$$

$$= x + \int \sec^2 x \, dx$$

$$= x + \tan x + c$$

13. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0.$

Solution

Given: $y = \sin(\sin x)$

To prove: $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos - x = 0$

That is,

$$y = \sin(\sin x)$$

$$\frac{dy}{dx} = \cos(\sin x)\cos x$$

$$\frac{d^2y}{dx^2} = \cos(\sin x)(-\sin x) + \cos^2 x(-\sin(\sin x)$$

$$\frac{d^2y}{dx^2} = -\sin x \cos(\sin x) - \cos^2 x \sin(\sin x)$$

$$\text{LHS} = \frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x$$

$= -\sin x \cos (\sin x) - \cos^2 x \sin (\sin x) + \tan x \cos x \cos (\sin x) + \cos^2 x \sin x (\sin x)$

$= -\sin x \cos(\sin x) + \frac{\sin x}{\cos x} \cos x \cos(\sin x)$

$= -\sin x \cos (\sin x) + \sin x \cos (\sin x)$

$= 0 =$ RHS

Hence, proved.

14. Find the particular solution of the differential equation $e^x \tan y dx + (2 - e^x)\sec^2 y dy = 0$; given that $y = \frac{\pi}{4}$ when $x = 0$.

Solution

We have

$$e^x \tan y dx + (2 - e^x)\sec^2 y dy = 0$$

Given: $y = \frac{\pi}{4}$ when $x = 0$.

$$e^x \tan y dx + (2 - e^x)\sec^2 y dy = 0$$

$$e^x \tan y dx = (e^x - 2)\sec^2 y dy$$

that is,

$$\frac{dy}{dx} = \frac{e^x \tan y}{e^x \sec^2 y - 2\sec^2 y}$$

$$\frac{dx}{dy} = \frac{e^x \sec^2 y - 2\sec^2 y}{e^x \tan y}$$

$$\frac{dx}{dy} = \frac{\sec^2 y}{\tan y} - \frac{2\sec^2 y}{\tan y} e^{-x}$$

$$\frac{dx}{dy} = \frac{\sec^2 y}{\tan y}(1 - 2e^{-x})$$

Now, $$\int \frac{\sec^2 y}{\tan y} dy = \int \frac{1}{1-2e^{-x}} dx$$

$$\tan y = t$$

$$\sec^2 y dy = dt$$

$$\int \frac{dt}{t} = \int \frac{e^x}{e^x - 2} dx$$

That is, $$e^x - 2 = u$$

$$e^x dx = du$$

$$\log t = \log u + \log c$$

$$\log(\tan y) = \log(e^x - 2)c$$

$$\tan y = c(e^x - 2)$$

Substituting $y = \frac{\pi}{4}$, $x = 0$, we get

$$\tan\frac{\pi}{4} = c(1-2)$$

That is, $c = -1$. Therefore,

$$\tan y = -(e^x - 2)$$

(OR)

Find the particular solution of the differential equation $\frac{dy}{dx} + 2y\tan x = \sin x,$ given that $y = 0$ when $x = \frac{\pi}{3}$.

Solution

Given: $y = 0$ when $x = \frac{\pi}{3}$.

To find the solution of the given differential equation:

$$\frac{dy}{dx} + 2y\tan x = \sin x$$

Integral function is

$$e^{\int 2\tan x} = e^{2\log\sec x}$$

$$= \sec^2 x$$

Now, $$y\sec^2 x = \int \sec^2 x \cdot \sin x \, dx + c$$

$$= \int \frac{\sin x}{\cos^2 x} dx + c$$

$$= \int \tan x \sec x \, dx + c$$

That is, $y\sec^2 x = \sec x + c$

15. Find the shortest distance between the lines $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$.

Solution

Given: $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$.

We have

$$\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$$

$$\vec{a}_1 = 4\hat{i} - \hat{j}$$

$$\vec{b}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$$

$$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$$

$$\vec{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{b}_2 = 2\hat{i} + 4\hat{j} - 5\hat{k}$$

The shortest distance between the lines is

$$\left| \frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

That is, $$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$$= \hat{i}(-10 + 12) - \hat{j}(-5 + 6) + \hat{k}(4 - 4)$$

$$= 2\hat{i} - \hat{j}$$

Therefore, the shortest distance is

$$\left| \frac{(-3\hat{i} + 0\hat{j} + 2j).(2\hat{i} - \hat{j})}{\sqrt{(2)^2 + (-1)^2}} \right| = \left| \frac{(-3 \times 2 + 0 + 0)}{\sqrt{4+1}} \right| = \left| \frac{-6}{\sqrt{5}} \right| = \frac{6}{\sqrt{5}}$$

16. Two numbers are selected at random (without replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X.

Solution

The first five positive integers are 1, 2, 3, 4, 5; we select two positive numbers in

$$5 \times 4 = 20 \text{ ways}$$

Among these two numbers, two numbers are selected at random.

Let us consider that X denote larger of the two numbers.

The total number of possible outcomes is 30.

- $P(X=2) = P(\text{larger number} = 2)$
  $= \{(1, 2) \text{ and } (2, 1)\}$
  $= \frac{2}{30}$
- $P(X=3) = P(\text{larger number} = 3)$
  $= \{(1, 3), (2, 3), (3, 2), (3, 1)\}$
  $= \frac{4}{30}$
- $P(X=4) = \text{P}(\text{larger number} = 4)$
  $= \{(1, 4), (2, 4), (3, 4), (4, 3), (4, 2), (4, 1)\}$
  $= \frac{6}{30}$
- $P(X=5) = P(\text{larger number} =5)$
  $= \{(1, 5), (2, 5), (3, 5), (4, 5), (5, 4), (5, 3), (5, 2), (5, 1)\}$
  $= \frac{8}{30}$

Therefore, the mean is

$$E(x) = 2\times\frac{2}{30} + 3\times\frac{4}{30} + 4\times\frac{6}{30} + 5\times\frac{8}{30}$$

$$= \frac{4+12+24+40}{30} = \frac{80}{30} = \frac{8}{3}$$

and the variance is

$$2^2\times\frac{2}{30} + 3^2\times\frac{4}{30} + 4^2\times\frac{6}{30} + 5^2\times\frac{8}{30} = \frac{8+36+96+200}{30} = \frac{340}{30}$$

That is, the variance is $\frac{34}{3}$ and the mean is $\frac{8}{3}$.

17. Using properties of determinants, prove that $\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx)$

Solution

To prove:-

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + zy + yx)$$

Now, $$xyz\begin{vmatrix} \frac{1}{x} & \frac{1}{x} & \frac{1}{x}+3 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{y} & \frac{1}{y}+3 & \frac{1}{y} \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$:

$$xyz\begin{vmatrix} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{y} & \frac{1}{y}+3 & \frac{1}{y} \end{vmatrix}$$

$$=(xyz)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{y}+3\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{y} & \frac{1}{y}+3 & \frac{1}{y} \end{vmatrix}$$

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$:

$$xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{y}+3 & -3 & -3 \\ \frac{1}{y} & 3 & 0 \end{vmatrix}$$

$$= (yz + zx + xy + 3xyz)\,(0 + 9)$$

$$= 9(3xyz + xy + zy + zx) = \text{RHS}$$

Hence the given matrix equation is proved.

18. Find the equations of the tangent and the normal to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$.

Solution

We have

$$16x^2 + 9y^2 = 145 \qquad (1)$$

Since (x_1, y_1) lies on the curve provided in Eq. (1), we have

$$16x_1^2 + 9y_1^2 = 145$$

$$16(2)^2 + 9y_1^2 = 145$$

$$16 \times 4 + 9y_1^2 = 145$$

$$64 + 9y_1^2 = 145$$

$$9y_1^2 = 145 - 64$$

$$9y_1^2 = 81$$

$$y_1^2 = 9$$

$$y_1 = 3$$

That is, $(x_1, y_1) = (2, 3)$. Therefore,

$$16x^2 + 9y^2 = 145$$

Differentiating w.r.t. x, we get

$$32x + 18y\frac{dy}{dx} = 0$$

$$18y\frac{dy}{dx} = -32x$$

$$\frac{dy}{dx} = \frac{-32x}{18y} \quad = \frac{-16x}{9y}$$

Now, $$\left(\frac{dy}{dx}\right)_{(2,3)} = \frac{-16\times 2}{9\times 3} \Rightarrow \frac{-32}{27}$$

Now, the equation of tangent is given by

$$(y - 3) = m(x - 2)$$

$$(y-3) = \frac{-32}{27}(x-2)$$

$$27y - 81 = -32x + 64$$

$$32x + 27y = 145$$

Now, the equation of normal is given by

$$(y - y_1) = \frac{-1}{m}(x - x_1)$$

$$(y-3) = \frac{27}{32}(x-2)$$

$$32y - 96 = 27x - 54$$

$$27x - 32y + 42 = 0$$

(OR)

Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is (a) strictly increasing; (b) strictly decreasing.

Solution

We have the function:

$$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$$

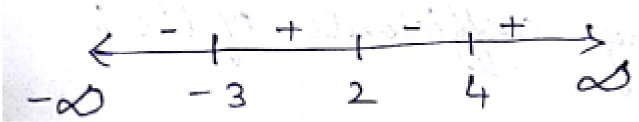
That is, $$f(x) = \frac{4x^3}{4} - 3x^2 - 10x + 24$$

$$f'(x) = 0$$

That is, $$x^3 - 3x^2 - 10x + 24 = 0$$

$$(x - 2)(x^2 - x - 12) = 0$$

$$(x - 2)(x + 3)(x - 4) = 0$$



That is, increasing in interval is

$$(-3, 2) \cup (4, \infty)$$

and the decreasing in interval is

$$(-\infty, 3) \cup (2, 4)$$

19. Find: $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$

Solution

We have

$$\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$$

Substituting $\sin x = t$, we get

$$\cos x dx = dt$$

That is, $$\int \frac{2dt}{(1-t)(1+t^2)}$$

$$\frac{2}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+c}{1+t^2}$$

$$2 = A(1 + t^2) + (Bt + c)(1 - t)$$

Substituting $1 - t = 0$, we get

$$t = 1$$

$$2 = A(2)$$

$$A = 1$$

On comparing coefficients of t^2 and t, we get

$$t^2 \rightarrow A + (-B) = 0$$

Now, $B = A$, that is, $B = 1$; therefore,

$$t \rightarrow B - C = 0$$

$$B = C = 1$$

$$\int \left(\frac{1}{1-t} + \frac{t+1}{t^2+1} \right) dt$$

$$\frac{\log(1-t)}{-1} + \int \frac{t}{t^2+1} dt + \int \frac{1}{t^2+1} dt$$

$$-\log(1-\sin x)+\frac{1}{2}\log(t^2+1)+\tan^{-1}t+c$$

$$-\log(1-\sin x)+\frac{1}{2}\log(\sin^2 x+1)+\tan^{-1}(\sin x)+c$$

20. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die?

Solution

Let us assume that E_1 be the event that the girl gets 1 or 2 on the roll; therefore,

$$P(E_1)\Rightarrow\frac{2}{6}=\frac{1}{3}$$

Let us assume that E_2 be the event that the girl gets 3, 4, 5 or 6 on the roll; therefore,

$$P(E_2)=\frac{4}{6}\ =\frac{2}{3}$$

Let us consider that A be event that she obtained exactly one tails.

If she tossed a coin 3 times and exactly 1 tail is shown, then

[HTH, HHT, THH] = 3

$$P(A/E_1)=\frac{3}{8}$$

If the girl tossed a coin only once and exactly 1 is shown, then

$$P(A/E_2)=\frac{1}{2}$$

$$P(E_2/A)=\frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$=\left(\frac{\frac{1}{2}\times\frac{2}{3}}{\left(\frac{1}{2}\times\frac{2}{3}+\frac{3}{8}\times\frac{1}{3}\right)}\right)=\frac{\left(\frac{1}{3}\right)}{\left(\frac{1}{3}+\frac{1}{8}\right)}=\frac{1/3}{11/24}\ =\frac{8}{11}$$

21. Let $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}\cdot\vec{a}=21$.

Solution

We have

$$\vec{a}=4\hat{i}+5\hat{j}-\hat{k}$$

$$\vec{b}=\hat{i}-4\hat{j}+5\hat{k}$$

$$\vec{c}=3\hat{i}+\hat{j}-\hat{k}$$

To find the vector $\vec{d}$ such that $\vec{d}\cdot\vec{c}=0$, $\vec{d}\cdot\vec{b}=0$ and $\vec{d}\cdot\vec{a}=21$, let us consider

$$\vec{d}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} + \hat{j} - \hat{k}) = 0$$

Now, $$3x + y - z = 0 \quad (1)$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} - 4\hat{j} + 5\hat{k}) = 0$$

$$x - 4y + 5z = 0 \quad (2)$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).(4\hat{i} + 5\hat{j} - \hat{k}) = 21$$

$$4x + 5y - z = 21 \quad (3)$$

Now, Eq. (1) × 4 + Eq. (2) yields the following:

$12x + 4y - 4z = 0$

$x - 4y + 5z = 0$

$13x + z = 0$

Also, Eq. (2) × 5 + Eq. (3) × 4 yields the following:

$5x - 20y + 25z = 0$

$16x + 20y - 4z = 84$

$21x + 21z = 84$

Now, $$x + y = 4 \quad (5)$$

Now, Eq. (4) – Eq. (5) yields the following:

$$12x = -4$$

$$x = \frac{-4}{12} = \frac{-1}{3}$$

$z = 4 - x$

$$z = 4 + \frac{1}{3} \Rightarrow \frac{4 \times 3 + 1}{3} \Rightarrow \frac{12 + 1}{3}$$

$$= \frac{13}{3}$$

Substituting x and z in Eq. (1), we get

$$3x + y - z = 0$$

$$3\left(\frac{-1}{3}\right) + y - \frac{13}{3} = 0$$

$$\frac{-3}{3} + y \frac{-13}{3} = 0$$

$$\frac{-16}{3} + y = 0$$

$$y = \frac{16}{3}$$

Therefore, $$\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$$

That is, $$\vec{d} = \frac{-1}{3}\hat{i} + \frac{16}{3}\hat{j} + \frac{13}{3}\hat{k}$$

22. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question?

Solution

Let us assume that the length, width and height of the open tank be x, x and y units.

The volume is x^2y.

The total surface area is

$$x^2 + 4xy$$

Now, $$S = x^2 + 4x\left(\frac{1}{x^2}\right)$$

Therefore, $$\frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

$$2x^3 = 4V$$

$$2x^3 = 4(x^2y)$$

$$x = 2y$$

That is, $$\frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3} = 2 + \frac{8V}{8y^3} = 2 + \frac{V}{y^3} > 0$$

Hence, S is minimum, where $x = 2y$, that is, the depth or the height of the tank is half of its width.

23. If $(x^2 + y^2)^2 = xy$, find $\frac{dy}{dx}$.

Solution

We have

$$(x^2 + y^2)^2 = xy$$

$$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = \frac{xdy}{dx} + y$$

$$4x(x^2 + y^2) + 4y(x^2 + y^2)\frac{dy}{dx} = \frac{xdy}{dx} + y$$

$$4x(x^2 + y^2) - y = \frac{dy}{dx}(x - 4y(x^2 + y^2))$$

Therefore, $$\frac{dy}{dx} = \frac{4x(x^2 + y^2) - y}{x - 4y(x^2 + y^2)}$$

(OR)

If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

Solution

Given: $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$.

Now, $x = a(2\theta - \sin 2\theta)$

Therefore, $$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta)$$

$$y = a(1 - \cos 2\theta)$$

$$\frac{dy}{d\theta} = a(0 + 2\sin 2\theta)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2a\sin 2\theta}{2a(1-\cos 2\theta)}$$

Also, $$\left(\frac{dy}{dx}\right)_{(\theta=\pi/3)} = \frac{\sin(2\pi/3)}{1-\cos(2\pi/3)}$$

$$= \frac{\sin(\pi - \pi/3)}{1-\cos(\pi-\pi/3)} = \frac{\sin \pi/3}{1+\cos \pi/3} = \frac{\sqrt{3}/2}{1+(1/2)} = \frac{1}{\sqrt{3}}$$

24. Evaluate: $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$

Solution

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx = \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9[1 - (\sin x - \cos x)^2]}$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 9(\sin x - \cos x)^2} dx$$

Now, $\sin x - \cos x = t$

That is, $(\cos x + \sin x)dx = dt$

Therefore, $$\int_{-1}^{0} \frac{dt}{25 - 9t^2} = \int_{-1}^{0} \frac{1}{9(25/9 - t^2)} dt$$

$$= \frac{1}{9}\int_{1}^{0} \frac{dt}{(5/3)^2 - t^2}$$

$$= \frac{1}{9}\left[\frac{1}{2\times 5/3}\log\left[\frac{5/3+t}{5/3-t}\right]\right]_{-1}^{0}$$

$$= \frac{1}{9}\times\frac{1}{10/3}\left(\log\frac{(5/3)}{(5/3)}\right) - \log\frac{(2/3)}{8/3}$$

$$= \frac{1}{30}(\log 1 - \log 1/4)$$

$$\Rightarrow \frac{1}{30}\log 4 = \frac{1}{15}\log 2$$

(OR)

Evaluate: $\int_1^3 (x^2 + 3x + e^x)dx$ as the limit of the sum.

Solution

We have

$$\int_1^3 (x^2+3x+e^x)dx \;= \lim_{h\to 0} h[f(1)+f(1+h)+f(1+2h)+\cdots+f(1+2(n-1)b)]$$

$$= \lim_{h\to 0} h[(1+3+e)+(1+h)^2+3(1+h)+e^{1+h})+((1+2h)^2+3(1+2h)+e^{1+2h})+\cdots]$$

$$= \lim_{h\to 0} h[4+e+(1+h^2+2h+3+3h+e^{1+h})+(1+4h^2+4h+3+6h+e^{1+2h})+\cdots]$$

$$= \lim_{h\to 0} h[4+e+(4+h^2+5h+e^{1+h})+(4+4h^2+10h+e^{1+2h})+\cdots]$$

$$= \lim_{h\to 0} h[4h+e(1+e^h+e^{2h}+\cdots)+h^2(1^2+2^2+\cdots)+5h(1+2+\cdots)]$$

$$= \lim_{h\to 0} h\left[4n+e\left(\frac{1(e^{hn}-1)}{e^h-1}\right)+h^2\left(\frac{n(n-1)(2n-1)}{6}\right)+5h\left(\frac{h(n-1)}{2}\right)\right]$$

$$= \lim_{h\to 0} h\left[4n+e\left(\frac{e^{nh}-1}{e^h-1}\right)+\frac{h^2n^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{5hn^2}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= \lim_{h\to 0} h\left[4n+e\left(\frac{e^{nh}-1}{e^h-1}\right)+\frac{h^2n^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{5}{2}\times n^2\times\frac{2}{n^2}\left(1-\frac{1}{n}\right)\right]$$

$$= \lim_{h\to 0} \frac{2}{n}\left[4n+e\left(\frac{e^{\frac{2n}{h}}-1}{e^{2/n-1}}\right)+\frac{4}{n^2}-\frac{n^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{5}{2}\times n^2\times\frac{2}{n}\left(1-\frac{1}{n}\right)\right]$$

$$= \lim_{h\to 0} \frac{2}{n}\left[4n+e\left(\frac{e^2-1}{e^{2/n}-1}\right)+\frac{4n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+n\times 5\left(1-\frac{1}{n}\right)\right]$$

$$= \lim_{h\to\infty} 2\left[4+\frac{e}{h}\frac{(e^2-1)}{e^{2/n}-1}+\frac{2}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+5\left(1-\frac{1}{n}\right)\right]$$

$$= 8+e(e^2-1)+\frac{4}{3}+5$$

$$= \frac{24+4+15}{3}+e(e^2-1)$$

$$= \frac{43}{3}+e(e^2-1)$$

25. A factory manufactures two types of screws A and B, each type requiring the use of two machines, an automatic and a hand-operated. It takes 4 min on the automatic and 6 min on the hand-operated machines to manufacture a packet of screws A while it takes 6 min on the automatic and 3 min on the hand-operated machine to manufacture a packet of screws B. Each machine is available for at most 4 hr on any day. The manufacturer can sell a packet of screws A at a profit of 70 paise and screws B at a profit of Rupee 1. Assuming that he can sell all the screws he manufactures, how many packets of each type should the factory owner produce in a day in order to maximize his profit? Formulate the above LPP and solve it graphically and find the maximum profit.

Solution

Let us consider that the number of package of screw A be x; the number of packages of screw B be y.

Article	**Number**	**Machine A**	**Machine B**	**Profit**
Screw A	x	4 min	6 min	To paise = Rs. 0.7
Screw B	y	6 min	3 min	Rupee 1
Max available time		4 h = 240 min	4 h = 240 min	

- **Automated machine:** Works for screw A→4 min; Works on screw B→6 min

  Therefore,

$$4x + 6y \leq 240$$

$$2x + 3y \leq 120 \qquad (x, y \geq 0)$$

- **Hand operated machine:** Works on screw A→6 min; Works on screw B→3 min. Therefore,

$$6x + 3y \leq 240$$

$$2x + y \leq 80 \qquad (x, y \geq 0)$$

Now, maximum $Z = 0.7x + y$

$$2x + 3y \leq 120$$

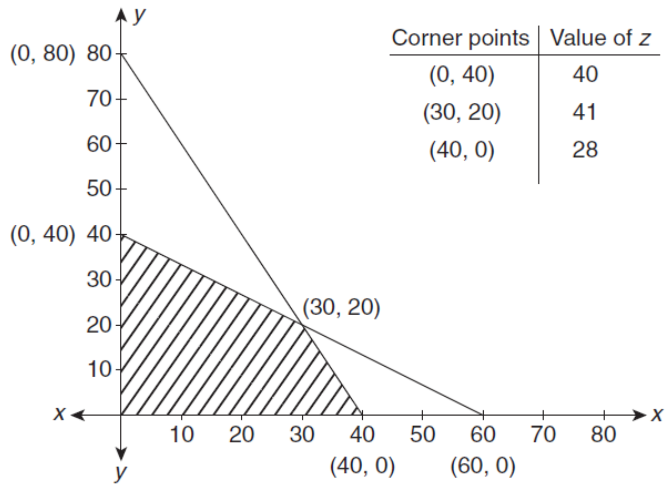
$$2x + y \leq 80 \qquad (x, y \geq 0)$$

$$2x + 3y \leq 120$$

x	0	60
y	40	0

$$2x + y \leq 80$$

x	0	40
y	80	0



Corner Points	Value of z
(0,40)	40
(30,20)	41
(40,0)	28

Therefore, the profit is maximum if the company produces the following:

- 30 packages of screw A
- 20 packages of screw B

Therefore, the maximum profit is Rs.41.

26. Let $A = \{x \in \mathbb{Z} : 0 \le x \le 12\}$. Show that $\mathbb{R}$ = {(*a*, *b*) : *a*, *b* $\in$ *A*, |*a* – *b*| is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2].

(OR)

Solution

We have

$$\mathbb{R} = \{(a,b) : a,b \in A, |a-b| \text{ is divisible by } 4\}$$

- **Reflexivity:** For any $a \in A$: $|a - a| = 0$, which is divisible by 4 $(a,a) \in \mathbb{R}$. Thus, $\mathbb{R}$ is reflexive.
- **Symmetry:** Let $(a,b) \in \mathbb{R}$: $|a - b|$ is divisible by 4; therefore, $\mathbb{R}$ is symmetry.
- **Transitive:** Let $(a, b) \in \mathbb{R}$ and $(b, c) \in \mathbb{R}$: $|a - b|$ is divisible by 4.

$$|a - b| = 4\lambda$$

$$a - b = \pm 4\lambda \quad (1)$$

Now, $|b - c|$ is divisible by 4.

$$|b - c| = 4\mu$$

$$b - c = \pm 4\mu \quad (2)$$

Adding Eqs. (1) and (2), we get

$$a - b + b - c = \pm 4(\lambda + \mu)$$

$$a - c = \pm 4(\lambda + \mu)$$

$$\Rightarrow \ (a, c) \in \mathbb{R}$$

Therefore, it is transitive.

Hence, $\mathbb{R}$ is reflexive, symmetric and transitive; thus, it is an equivalence relation.

Let x be an element of A such that $(x,1) \in \mathbb{R}$, then $|x - 1|$ is divisible by 4.

$$x - 1 = 0, 4, 8, 12$$

$$x = 1, 5, 9$$

Hence, the set of all element of A which are related to 1 in {1, 5, 9}.

(OR)

Show that the function f: $\mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{x^2 + 1}, \forall x \in \mathbb{R}$ is neither one–one nor onto. Also, if g : $\mathbb{R} \to \mathbb{R}$ is defined as $g(x) = 2x - 1$, find $f \circ g(x)$.

Solution

We have

$$f(x) = \frac{x}{x^2 + 1}$$

- For one–one function: $f(x) = f(y)$; that is,

$$\frac{x}{x^2 + 1} = \frac{y}{y^2 + 1}$$

$$xy^2 + x = yx^2 + y$$

$$xy(y - x) = y - x$$

$$xy = 1$$

$$x = \frac{1}{y}$$

$$x \neq y$$

Therefore, it is not one–one function.

- For onto: $f(x) = y$

$$\frac{x}{x^2 + 1} = y$$

$$x = yx^2 + y$$

$$x^2y + y - x = 0$$

That is, x cannot be expressed in y; hence, it is not onto function.

As $g(x) = 2x - 1$, we have

$$f \circ g(x) = f[(g(x)] \Rightarrow f(2x - 1) \ = \frac{2x - 1}{(2x - 1)^2 + 1} \ = \frac{2x - 1}{4x^2 - 4x + 2}$$

27. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

Solution

We have

$$x^2 + y^2 = 32$$

$$x^2 + y^2 = (\sqrt{32})^2 = (4\sqrt{2})^2$$

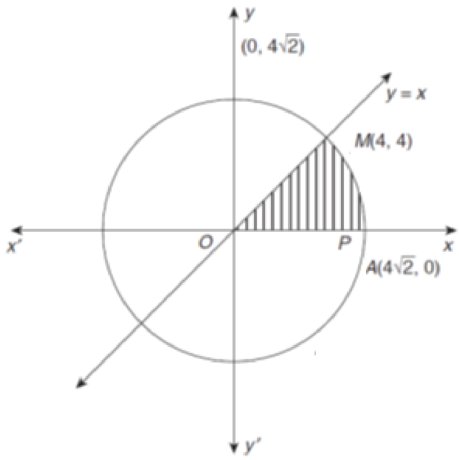
For coordinate of M, substitute $y = x$ in $x^2 + y^2 = 32$; therefore,

$$2x^2 = 32$$

$$x^2 = 16$$

$$x = \pm 4$$

that is, $M(4, 4)$.



The required area is the area of the shaded region. That is:

$$\text{Area of } OMA = \text{Area of } OMP + \text{area } MPA$$

$$= \int_0^4 y_1 dx + \int_4^{4\sqrt{2}} y_2 dx$$

$$= \int_0^4 x dx + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2 dx}$$

$$= \left(\frac{x^2}{2}\right)_0^4 + \left[\frac{x}{2}\sqrt{(4\sqrt{2})^2 - x^2} + \frac{(4\sqrt{2})^2}{2}\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$$

$$= \frac{16}{2} + \left(\frac{4\sqrt{2}}{2}\sqrt{(4\sqrt{2})^2 - (4\sqrt{2})^2} + \frac{3^2}{2}\sin^{-1} 1\right)$$

$$= \left(\frac{4}{2}\sqrt{(4\sqrt{2})^2 - 4^2} + \frac{3^2}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right)$$

$$= 8 + \left(2\sqrt{2}(0) + 16 \times \frac{\pi}{2}\right) - \left(2 \times 4 + 16 \times \frac{\pi}{4}\right)$$

$$= 8 + 0 + 8\pi - 8 - 4\pi \;= 4\pi$$

28. If $A=\begin{bmatrix}2 & -3 & 5\\3 & 2 & -4\\1 & 1 & -2\end{bmatrix}$, find A^{-1}. Use it to solve the system of equations:

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Solution

We have

$$A=\begin{bmatrix}2 & -3 & 5\\3 & 2 & -4\\1 & 1 & -2\end{bmatrix}$$

That is, $|A| = 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$

$$|A| = 0 + 3(-2) + 5 \times 1$$
$$= 0 - 6 + 5 \neq 0$$

Now, $A_{11} = 0,\ A_{12} = 2;\ A_{13} = 1$

$$A_{21} = -1,\ A_{22} = -9;\ A_{23} = -5$$
$$A_{31} = 2;\ A_{32} = 23;\ A_{33} = 13$$

That is, $A_{ij}=\begin{bmatrix}0 & 2 & 1\\-1 & -9 & -5\\2 & 23 & 13\end{bmatrix}$

Therefore, $\text{Adj } A=[A_{ij}]^{\text{T}}=\begin{bmatrix}0 & 2 & 1\\-1 & -9 & -5\\2 & 23 & 13\end{bmatrix}^{\text{T}}=\begin{bmatrix}0 & -1 & 2\\2 & -9 & 23\\1 & -5 & 13\end{bmatrix}$

$$A^{-1}=\frac{\text{Adj } A}{|A|}=\frac{-1}{1}\begin{bmatrix}0 & -1 & 2\\2 & -9 & 23\\1 & -5 & 13\end{bmatrix}=\begin{bmatrix}0 & 1 & -2\\-2 & 9 & -23\\-1 & 5 & -13\end{bmatrix}$$

Now, $\begin{bmatrix}2 & -3 & 5\\3 & 2 & -4\\1 & 1 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}11\\-5\\-3\end{bmatrix}$

That is, $AX = B$

$$X = A^{-1}B$$
$$=\begin{bmatrix}0 & 1 & -2\\-2 & 9 & -23\\-1 & 5 & -13\end{bmatrix}\begin{bmatrix}11\\-5\\-3\end{bmatrix}$$

That is, $\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}0 & -5 & +6\\-22 & -45 & +69\\-11 & -25 & +39\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

Therefore, $x = 1$, $y = 2$ and $z = 3$.

(OR)

Using elementary row transformations, find the inverse of the matrix:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}.$

Solution

Here, we have

$$A = IA \qquad \text{(inverse of matrix)}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1$:

$R_3 \to R_3 + 2R_1$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 3R_3$:

$R_2 \to R_2 - R_3$:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -3 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

29. Find the distance of the point (−1, −5, −10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

Solution

The eequation of line is given by

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$$

And the equation of the plane is given by

$$\vec{r}(\hat{i} - \hat{j} + \hat{k}) = 5$$

Now, the combined equation is

$$[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})](\hat{i} - \hat{j} + \hat{k}) = 5$$

$$[(2+3\lambda)\hat{i}+(-1+4\lambda)\hat{j}+(2+2\lambda)\hat{k}].(\hat{i}-\hat{j}+\hat{k})=5$$

$$2+3\lambda-(-1+4\lambda)+2+2\lambda=5$$

$$\lambda+5=5$$

$$\lambda=0$$

Therefore, the equation of line is

$$\vec{r}=(2\hat{i}-\hat{j}+2\hat{k})+0(3\hat{i}+4\hat{j}+2\hat{k})$$

$$\vec{r}=(2\hat{i}-\hat{j}+2\hat{k})$$

Let the point of intersection be (x, y, z); thus,

$$\vec{r}=x\hat{i}+y\hat{j}+2\hat{k}$$

Therefore, $x = 2$, $y = -1$ and $z = 2$

Hence, the point of intersection is (2, −1, 2).

Distance between of the given point from the point of intersection of the given line and the plane is

[(2, −1, 2) and (−1, −5, −10)]

That is, $\sqrt{(-1-2)^2+(-5+1)^2+(-10-2)^2}=\sqrt{9+16+144}\Rightarrow\sqrt{169}=13$